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CONSTRAINED DECONVOLUTION BY EXPLICIT CORRECTION OF FOURIER COMPONENTS OF THE UNKNOWN DISTRIBUTION

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SUMMARY: The physical constraints were applied in the problem of deconvolution by explicitly correcting noise affected Fourier components of the unknown distribution. The method gives the constrained estimate optimal in the quadratic sense, i.e. the estimate closest to the exact solution in the Euclidean space of solutions. The properties of the method were theoretically examined and some practical applications in the astronomical spectroscopy have been effected. The method is compared with the similar Fourier spectrum extrapolation procedures; as a consequence application of the method is recommended particularly in the case of low signal-to-noise measurements.

1. INTRODUCTION

1.1 Mathematical model of convolution

In the real world the observed physical phenomenon is ordinarily a superposition of two or more different phenomena. Without loosing in generality, we can consider two physical phenomena (that can be a complex physical phenomenon itself) that are quantitatively represented by the distributions $o(\mathbf{\Lambda})$ and $s(\mathbf{\Lambda})$ with respect to the frame of the independent variables $\mathbf{\Lambda}$. For simplicity, in this paper we treat one-dimensional physical quantities (with respect to an independent variable Λ), although most of the concepts may readily be extended to multiple dimensions.

Very often this superposition of two physical phenomena $o(\Lambda)$ and $s(\Lambda)$ can be represented by the mathematical model of convolution:

$$x(\Lambda) = \int_{-\infty}^{\infty} o(\Lambda')s(\Lambda - \Lambda')d\Lambda', \qquad (1)$$

i.e. the new distribution $x(\Lambda)$ is a weighted integral of the distribution $o(\Lambda)$, the distribution $s(\Lambda)$ supplying the required weight and being slid along $o(\Lambda)$ according to the displacement specified by Λ'.

In practice the distribution $x(\Lambda)$ can be obtained only in the finite interval $\mathcal{L}$ of an independent variable Λ, and is discretized by the detector, measurement machine or computer and, when the discretization is done equidistantly, the integral equation (1) is usually represented as:

$$x(\lambda) = \sum_{\lambda' \in \mathcal{L}} o(\lambda')s(\lambda - \lambda') \qquad \lambda \in \mathcal{L}, \qquad (2)$$

where the set $\mathcal{L}$ is defined as:

$$\mathcal{L} = \{\lambda | \lambda = 0, 1, ..., N-1\}$$

and N is the number of the measurements.

Considering also the effect of the random noise $n(\lambda)$, which is inevitably present in all physical measurements, the mathematical model representing our data is:

$$i(\lambda) = x(\lambda) + n(\lambda) = \sum_{\lambda' \in \mathcal{L}} o(\lambda') s(\lambda - \lambda') + n(\lambda). \quad (3)$$

The mathematical treatment of the problem can be further simplified by introducing for a given distribution $f(\lambda)$ of the space coordinate λ its Fourier transform $F(\omega)$ of spatial frequency (radians/length) ω defined as:

$$F(\omega) = N^{-1} \sum_{\lambda \in \mathcal{L}} f(\lambda) \exp\left(-\imath\omega\lambda \frac{2\pi}{N}\right) \quad (4)$$

and its pair:

$$f(\lambda) = N \sum_{\omega \in \mathcal{W}} F(\omega) \exp\left(+\imath\omega\lambda \frac{2\pi}{N}\right) \quad (5)$$

where $\imath$ denotes the imaginary unity, the set $\mathcal{W}$ is defined as:

$$\mathcal{W} = \{\omega | \omega = -\Omega_N, ..., -1, 0, 1, ...\Omega_N - 1\}$$

and Ω_N is the Nyquist frequency.

For the Fourier transform pairs:

$$o(\lambda), O(\omega);\ s(\lambda), S(\omega);\ n(\lambda), N(\omega)$$

the relation (3) can be expressed in the Fourier domain:

$$I(\omega) = O(\omega) \cdot S(\omega) + N(\omega) \qquad \omega \in \mathcal{W}. \quad (6)$$

To illustrate the convolution phenomenon described by the Eqs. (1) and (6), as a test example for the deconvolution methods, we focus our attention on a spectroscopical application of the described mathematical formalism. With numerically generated object spectrum $o(\lambda)$ shown in Fig. (1), the response function $s(\lambda)$ (Fig. 2. top) and one realization of computer generated random noise, the observed spectrum $i(\lambda)$ appears as it is shown in Fig. 2. (bottom). In practice the distribution $o(\lambda)$ is not known and only the general prior information about it (as positivity, smoothness etc.) is supposed to be known.

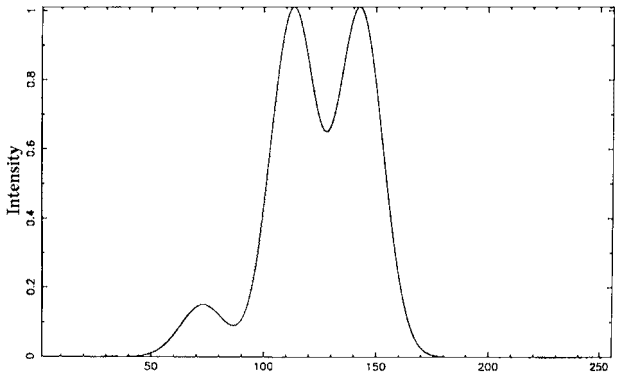

Fig. 1. *Object spectrum.*

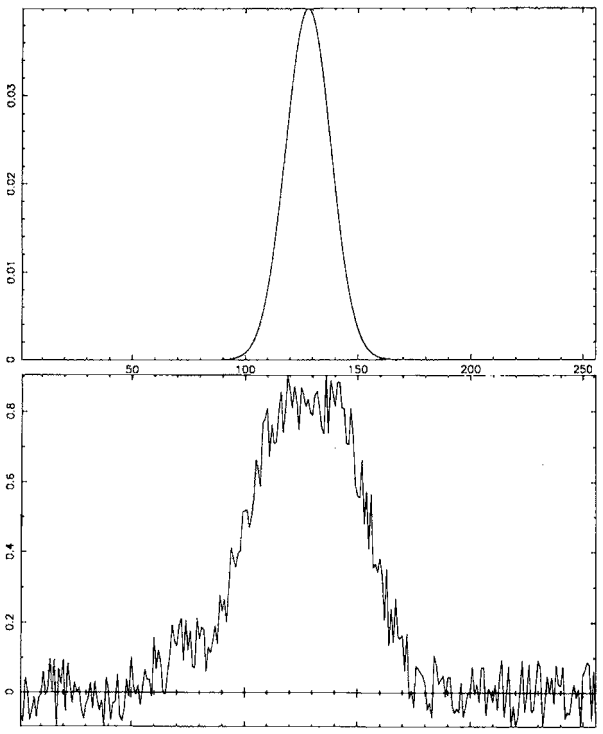

Fig. 2. *Response function (top) and "observed" spectrum (bottom).*

Considering that we have measurements $i(\lambda)$, and that the distribution $s(\lambda)$ is *a priori* known, our aim is to gain meaningful information about a phenomenon $o(\lambda)$ that is not a directly physically observable. This is, in fact, the task of the deconvolution.

1.2 Problem of deconvolution

The operation of deconvolution appears in many astrophysical problems (for an overview of the spectroscopical applications see Gray, 1976) and in particular represents an idealization of the problem of "unfolding data" through a known instrumental response function. The spectrometer completely obliterates the information at all Fourier frequencies beyond some finite cutoff Ω_E. This is specifically true at dispersive optical spectrometers, where the aperture determines Ω_E. Similar considerations prevail in Fourier interferometer, where the maximum path difference determines Ω_E.

Although the equation (6) is readily solved for $O(\omega)$ in the absence of the noise, the associated inversion problems, when viewed in a strict mathematical sense, are unstable against small perturbations $N(\omega)$ in the data function and often possess non-unique solutions. Thus, the problem of the deconvolution is fundamentally ill-posed. Also, as a consequence of the fact that the noise is not affected by the convo-

lution effects, the measured data function cannot provide sufficient information on the high frequency components of the solution.

To regularize the solutions and also to obtain some degree of super-resolution the extra information is needed. Thus, deconvolution is, in fact, an art of introducing the prior knowledge into the inversion of the equation (3).

Here we focus our attention on the methods working explicitly in the Fourier domain. The simple mathematical form of the equation (6) allows us to produce efficient and fast algorithms. Further advantage of Fourier representation is that it provides a way of exhibiting the data in the form more convenient to apply some constraints. Indeed, the Fourier transform of an absolutely integrable function is known to tend to zero as $\omega \to \infty$ and the smoother the function $o(\lambda)$ the faster its Fourier transform falls. Thus, the condition that the $O(\omega)$ identically equals zero outside a finite interval $|\omega| \leq \Omega_S$ appears to be natural.

It follows that, for example, the prior knowledge of signal cutoff Ω_S (that is stronger form of prior knowledge than smoothness) is easily implemented in this approach.

When comparing different deconvolution methods their fundamental properties should be taken into account:

a) Quantity of *a priori* information that can be implemented. For all correctly founded methods this property determines the quality of deconvolution.

b) Speed, that determines the quantity of data that can be processed in the unity of time.

c) Availability of *a priori* information used in the method. That information can be easily or hardly obtained, more or less reliable, so this property influences both a) and b). In this sense general *a priori* information as positivity, smoothness, finite extent etc. is more desirable.

d) Accommodability of the method. A number of deconvolution methods is developed for the intended purpose. These methods work best only for this purpose. But it is very desirable if a method can be accommodated for the different problems that could arise.

Very exhaustive discussions of the deconvolution methods in image enhancement and restoration problems can be found in Frieden (1979), also Narayan and Nityananda (1986) and, for the spectroscopical applications, in Jansson (1985).

2. FOURIER SPECTRUM EXTRAPOLATION METHODS

In the Fourier spectrum extrapolation approach, only the "most certain" components of the inverse filtered observed signal are used in order to obtain the non-constrained estimate $t(\lambda)$. The measured signal $i(\lambda)$ is filtered by a perfect low-pass filter that eliminates all frequencies beyond some truncation frequency Ω_E while leaving components at lower frequencies unchanged. So, the non-constrained estimate $t(\lambda)$ has the Fourier components $T(\omega)$:

$$T(\omega) = \begin{cases} I(\omega)/S(\omega) & |\omega| \leq \Omega_E \\ 0 & |\omega| > \Omega_E \end{cases} .$$

To ensure physically acceptable restored object, the constrained estimate $r_e(\lambda)$ can be represented as:

$$r_e(\lambda) = t(\lambda) + e(\lambda),$$

where $e(\lambda)$ is the bandwidth-extension function with non-zero Fourier components $E(\omega)$ in the

$$\Omega_E \leq |\omega| \leq \Omega_S$$

Fourier frequency interval.

Howard (1985) determined the Fourier components $E(\omega)$ by the minimization of the quadratic distance:

$$\sum_{\lambda \in \mathcal{L}} \Big\{ r_e(\lambda) \cdot h[-r_e(\lambda)] \Big\}^2, \tag{7}$$

where α is a variable parameter, introduced in order to allow the analytic differentation of the Eq. (7) and the function $h[x]$:

$$h[x] = \Big\{ 1 + \exp\big(x/\alpha\big) \Big\}^{-1}$$

approaches the unit step function as a limit:

$$\lim_{\alpha \to 0} h[x] = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases} .$$

In the approach, based on the Burg's entropy distance

$$\sum_{\lambda \in \mathcal{L}} \ln r_e(\lambda), \tag{8}$$

(Frieden, 1979, Komesaroff et al. 1981) use is made of the possibility of extrapolating an autocorrelation sequence from $2 \cdot \Omega_E + 1$ "known" autocorrelations. This approach originates from the fact that a positive object can be represented as the inverse Fourier transform of an autocorrelation function.

Minimizing the distances (7) or (8) one obtains the resulting equations for $E(\omega)$ in the closed and simple form. However, the main algorithmical difficulty of the later approach is that some other constraints in the data space (as finite extent, lower and upper bounds) cannot be easily implemented.

3. FOURIER SPECTRUM CORRECTION METHOD

3.1 Critical review of the Fourier spectrum extrapolation concept

Generally, although for the high signal-to-noise ratios the Fourier spectrum extrapolation methods give a noticeably improvement in a noise suppression and resolution of the restored spectrum, it should be noted that the concept of Fourier spectrum extrapolation suffers from the following drawbacks:

a) The noise accumulated in the retained low frequency Fourier components affects the accuracy of the extrapolation, so very far extrapolations are forbidden. For this reason the truncation frequency Ω_E should be chosen sufficiently large (as close as possible to Ω_S). Here we are in the situation to extrapolate noisy data, which is a problematic task, generally.

b) In order to avoid serious noise accumulation in the retained Fourier components we obviously have to choose a sufficiently small cutoff Ω_E. In this manner we have to lose the Fourier components significantly higher than the noise level. Thus a large part of the information content of the data (Fourier components of the signal with $\Omega_E < |\omega| \leq \Omega_S$) is immediately thrown away. This is illustrated in Fig. 3. (top) where the Fourier spectrum extrapolation reconstruction is shown. In this reconstruction only five complex Fourier components of the signal ($\Omega_E = 5$) are used to perform the extrapolation in Fourier domain.

c) Obviously the previous reasons are contradictory in the sense of choosing cutoff Ω_E. Practically the tradeoff is difficult to make. The Frieden's (1979) optimum processing bandwidth is too broad for the Fourier spectrum extrapolation methods. Choosing the Frieden's cutoff the accumulated noise can only be amplified by an algorithm or, in the best case, no improvements can be obtained. Figure 3. (bottom) shows the reconstruction where eight complex Fourier components of the signal are used to perform the extrapolation in Fourier domain.

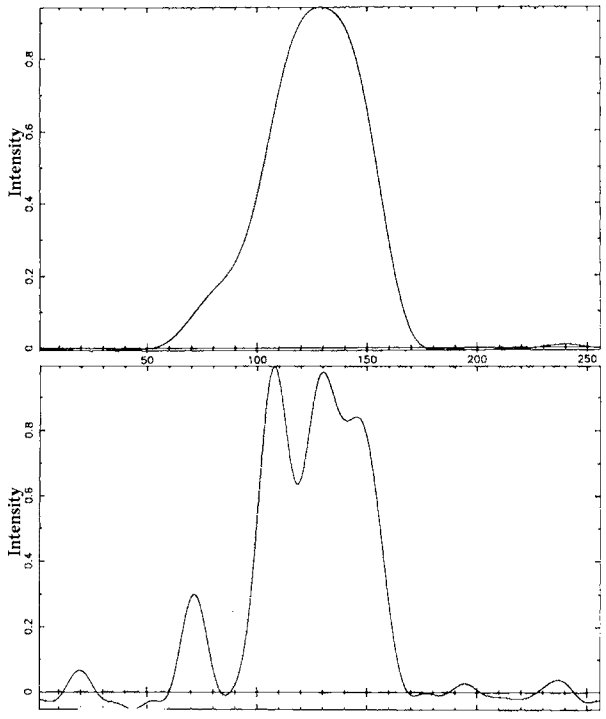

Fig. 3. *Reconstructions using the Fourier spectrum extrapolation approach. Fourier spectrum is truncated, and five (top) and eight (bottom) complex Fourier components of the signal were retained.*

For these reasons, in the Fourier spectrum extrapolation approach, the restored spectrum obtained by applying the physical constraints will generally depend on:

a) the initial approximation (determined by Ω_E); that is illustrated in Fig. 3. and

b) the noise; as it can be seen by comparing the reconstructions from Fig. 4. where the Howard's Fourier spectrum extrapolation algorithm is applied to noise-free data, and from Fig. 3. (bottom) where the same reconstruction procedure is applied to noisy data.

The similar behavior of the autocorrelation sequence extrapolation algorithm is experimentally examined by Komesaroff et al. (1981).

In principle the Fourier spectrum extrapolation concept is a natural approach for the kind of measurements in which a) some Fourier components of the signal are not measured (interferometry, Fourier spectroscopy) and b) the measured ones are obtained with an infinite (very high) signal-to-noise ratio. However the approach consistent with the presence of noise is to correct the Fourier components of the signal before attempting extrapolation. Biraud (1969) minimized the quadratic distance in Fourier space to correct the Fourier transform of the positive distribution by forcing it to be an autoconvolution product.

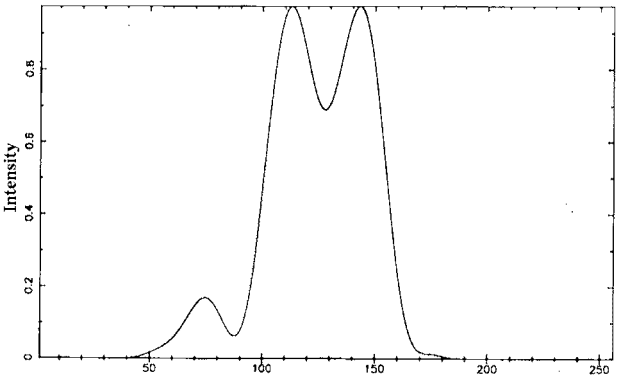

Fig. 4. *Noise free reconstruction using the Fourier spectrum extrapolation approach. Eight complex Fourier components of the signal were retained.*

3.2 Minimum quadratic distance approach

In order to correct the non-constrained restored object $r_{n-1}(\lambda)$, we express the constrained restored object $r_n(\lambda)$ by the relation:

$$r_n(\lambda) = r_{n-1}(\lambda) + c(\lambda)$$

where, according to (5), the unknown correction function $c(\lambda)$ can be expressed in terms of its Fourier components $C(\omega)$:

$$r_n(\lambda) = r_{n-1}(\lambda) + \sum_{\omega \in \mathcal{W}} C(\omega) \exp\left(+\imath\omega\lambda \frac{2\pi}{N}\right). \quad (9)$$

In the following, we show that the problem of determining the Fourier components $C(\omega)$ is also ill-posed so, in place of requiring that the constraints be satisfied exactly at a number of points equal to the number of undetermined coefficients we require that the sum of the square of $r_n(\lambda)$, over the subset of λ where the constraints on $r_n(\lambda)$ are not satisfied, be as small as possible. Thus the basic condition is of the form:

$$\sum_{\lambda\in\mathcal{L}'_n}\Big\{r_n(\lambda)\Big\}^2 = min, \tag{10}$$

where the summation is performed over the subset $\mathcal{L}'_n$:

$$\mathcal{L}'_n = \{\lambda|\lambda = \lambda_{\rm nc}\} \qquad \mathcal{L}'_n \subset \mathcal{L}$$

consisting of all outcomes $\lambda_{\rm nc}$ such that for $r_n(\lambda_{\rm nc})$ the constraints are *not* fulfilled.

Here we use the subscript n to denote that, generally, the set $\mathcal{L}'_n$ does not have the same elements over the progressing iterations. For some constraints, as the constraint of finite extent the set $\mathcal{L}'_n$ is constant.

In this environment the distributions $r_{n-1}(\lambda)$ and $r_n(\lambda)$ should be regarded as "less-constrained" and "more-constrained" respectively. For the non-constrained restored distribution $r_0(\lambda)$ we use the Wiener estimate that minimizes the quadratic distance:

$$\sum_{\lambda\in\mathcal{L}}\Big\{o(\lambda) - r_0(\lambda)\Big\}^2 = min. \tag{10'}$$

Figure (5) shows the test reconstruction using the Wiener optimal filter. That linear deconvolution method has a propensity to produce solutions that do not have good physical sense. The physical constraint of positivity is violated in a base-line (continuum) region and the reconstructed distribution contains negative values arising from noise.

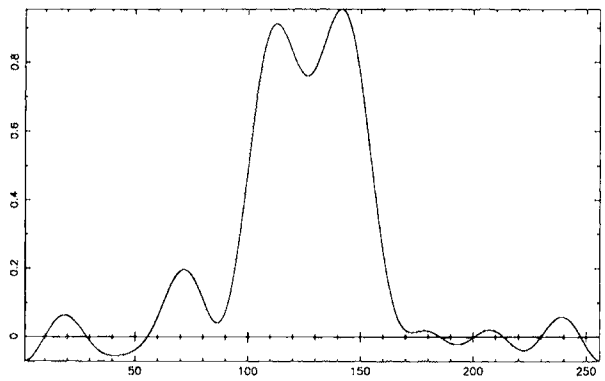

Fig. 5. *Reconstruction using Wiener-filter approach.*

In order to satisfy the requirement of positivity (10), the derivative of the left-hand member with respect to each parameter $C(\omega)$ must vanish, so inserting $r_n(\lambda)$ as given by (9) into (10) we must have:

$$\frac{\partial}{\partial C(\omega')}\sum_{\lambda\in\mathcal{L}'_n}\Big\{r_{n-1}(\lambda)+\sum_{\omega\in\mathcal{W}}C(\omega)\exp(+\imath\omega\lambda\frac{2\pi}{N})\Big\}^2 = 0,$$

where $\omega'\in\mathcal{W}$. These conditions take the form of:

$$\sum_{\lambda\in\mathcal{L}'_n} r_{n-1}(\lambda)\exp(+\imath\omega'\lambda\frac{2\pi}{N})+$$

$$+\sum_{\omega\in\mathcal{W}}C(\omega)\sum_{\lambda\in\mathcal{L}'_n}\exp(+\imath(\omega+\omega')\lambda\frac{2\pi}{N}) = 0. \tag{11}$$

Proceeding further, to apply the constraints in the Fourier domain, we mention that $O(\omega)$ is zero for $|\omega| > \Omega_S$ so we conclude, taking into account similar behaviour of $R_0(\omega)$, that $C(\omega)$ is also zero for $|\omega| > \Omega_S$. Further, we can assume that the surface of the measured distribution $i(\lambda)$ is not significantly affected by the noise, so because $S(0) = 1$, it follows that $C(0) = 0$.

In order to insert the above constraints on $C(\omega)$ into (11) we introduce the subset $\mathcal{W}'$:

$$\mathcal{W}' = \{\omega|\omega = \omega_{\rm nc}\} \qquad \mathcal{W}' \subset \mathcal{W}$$

consisting of all outcomes $\omega_{\rm nc}$ such that $C(\omega_{\rm nc})$ are not equal to zero. Explicitly:

$$\mathcal{W}' = \{\omega|\omega = -\Omega_S, ..., -1, 1, ...\Omega_S\} \qquad \Omega_S \in \Omega_N. \tag{12}$$

The Eq. (11) becomes:

$$\sum_{\lambda\in\mathcal{L}'_n} r_{n-1}(\lambda)\exp(+\imath\omega'\lambda\frac{2\pi}{N})+$$

$$+\sum_{\omega\in\mathcal{W}'}C(\omega)\sum_{\lambda\in\mathcal{L}'_n}\exp(+\imath(\omega+\omega')\lambda\frac{2\pi}{N}) = 0. \tag{13}$$

Further, by introducing the distribution:

$$p(\lambda) = \begin{cases} 1 & \lambda\in\mathcal{L}'_n \\ 0 & \lambda\notin\mathcal{L}'_n \end{cases} \tag{14}$$

one obtains:

$$\sum_{\lambda\in\mathcal{L}} z(\lambda)\exp(+\imath\omega'\lambda\frac{2\pi}{N})+$$

$$+\sum_{\omega\in\mathcal{W}'}C(\omega)\sum_{\lambda\in\mathcal{L}}p(\lambda)\exp(+\imath(\omega+\omega')\lambda\frac{2\pi}{N}) = 0, \tag{15}$$

where we have written:

$$z(\lambda) = p(\lambda)\cdot r_{n-1}(\lambda). \tag{16}$$

Because both $z(\lambda)$ and $p(\lambda)$ are real, we easily conclude, making use of the complex conjugate of Eq. (5), that the relation (15) can be written as:

$$Z^*(\omega') + \sum_{\omega\in\mathcal{W}'}C(\omega)\cdot\Pi^*(\omega+\omega') = 0. \tag{17}$$

Taking the complex conjugate of (17), and since $C(\omega)$ also should be a real function $C^*(\omega) = C(-\omega)$, we may write:

$$Z(\omega') + \sum_{\omega \in \mathcal{W}'} C(-\omega) \cdot \Pi(\omega + \omega') = 0$$

and finally, making use of the symmetry of $\mathcal{W}'$:

$$-Z(\omega') = \sum_{\omega \in \mathcal{W}'} C(\omega) \cdot \Pi(\omega' - \omega) \qquad \omega' \in \mathcal{W}'. \quad (18)$$

This is essentially the equation of convolution. Indeed, without constraints on $C(\omega)$, that implies $\mathcal{W}' = \mathcal{W}$, and comparing with equation (2) we see that the equation (18) represents the convolution in the Fourier domain, or multiplication in the λ domain:

$$-z(\lambda) = c(\lambda) \cdot p(\lambda)$$

so we see that the determination of $C(\omega)$ in the λ domain implies the division by the function $p(\lambda)$ defined by Eq. (14) that is, of course, the problem to be regularized.

Although the application of the constraints on $C(\omega)$ ensures an important degree of regularization, from the previous discussion we see the problem should be treated as fundamentally ill-posed.

4. METHODS FOR RESOLVING THE RELEVANT EQUATIONS

In order to obtain the coefficients $C(\omega)$ from the equation (18), where $Z(\omega)$ and $\Pi(\omega)$ are the Fourier transforms of (16) and (14) respectively, and finally to perform the constrained restoration using (9) and (10), several approaches are available:

The most attractive way is to reduce the number of equations (using only frequencies belonging to $\mathcal{W}'$) :

$$-Z(\omega') = \sum_{\omega=-\Omega_S}^{\Omega_S} C(\omega) \cdot \Pi(\omega' - \omega), \qquad (19)$$

where $\omega' = -\Omega_S, ..., -1, 0, 1, ..., \Omega_S$ and to resolve the convolution equation (19) with the constraint $C(0) = 0$, meaning that the deconvolved spectral lines contain the same area as their counterparts in the raw data. In this approach, choosing the convenient Ω_S (slightly higher than the actual cut-off) the effective algorithms, based on the Fast Fourier Transform can be developed.

Otherwise, in order to express the equation (18) in the matrix form and preserving the conjugate symmetry of $C(\omega)$, we may write:

$$-Z(\omega') = \sum_{\omega=1}^{\Omega_S} C(\omega) \cdot \Pi(\omega' - \omega) + \sum_{\omega=1}^{\Omega_S} C^*(\omega) \cdot \Pi(\omega' + \omega), \qquad (20)$$

where $\omega' = 1, 2, ..., \Omega_I$. Here we consider only the equations provided by $\omega' > 0$, since the equations provided by $\omega' < 0$ are redundant which can easily be approved by taking the complex conjugate of (20) and taking into account the conjugate symmetry of $Z(\omega)$ and $\Pi(\omega)$.

In the equation (20) we considered also that no significant information on $C(\omega)$ is contained in $Z(\omega)$ beyond some frequency, say $\Omega_I : \Omega_S \leq \Omega_I < \Omega_N$. It follows:

$$\mathbf{Z} = \mathbf{C} \star \mathbf{A} + \mathbf{C}^* \star \mathbf{B}, \qquad (21)$$

where $\star$ denotes the matrix product and the vectors $\mathbf{Z}$ and $\mathbf{C}$ have the elements:

$$\mathbf{Z} = \begin{pmatrix} Z(1) \\ Z(2) \\ \vdots \\ Z(\omega') \\ \vdots \\ Z(\Omega_I) \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} C(1) \\ C(2) \\ \vdots \\ C(\omega) \\ \vdots \\ C(\Omega_S) \end{pmatrix}$$

and matrices $\mathbf{A}$ and $\mathbf{B}$ are expressed explicitly as:

$$\mathbf{A} = \begin{pmatrix} \Pi(0) & \Pi^*(1) & \dots & \Pi^*(\omega-1) & \dots & \Pi(\Omega_S - 1) \\ \Pi(1) & \Pi(0) & \dots & \Pi^*(\omega-2) & \dots & \Pi^*(\Omega_S - 2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Pi(\omega-1) & \Pi(\omega-2) & \dots & \Pi(0) & \dots & \Pi^*(\Omega_S - \omega) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Pi(\Omega_I - 1) & \Pi(\Omega_I - 2) & \dots & \Pi(\Omega_I - \omega) & \dots & \Pi(\Omega_I - \Omega_S) \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} \Pi(2) & \Pi(3) & \dots & \Pi(\omega+1) & \dots & \Pi(\Omega_S + 1) \\ \Pi(3) & \Pi(4) & \dots & \Pi(\omega+2) & \dots & \Pi(\Omega + 2) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Pi(\omega+1) & \Pi(\omega+2) & \dots & \Pi(2\omega) & \dots & \Pi(\omega + \Omega_S) \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ \Pi(\Omega_I + 1) & \Pi(\Omega_I + 2) & \dots & \Pi(\Omega_I + \omega) & \dots & \Pi(\Omega_I + \Omega_S) \end{pmatrix}$$

But the main difficulty of the method lies in the fact that in (n-1)-*th* stage the domain $\mathcal{L}'_n$ is not yet known so we should use:

$$p'(\lambda) = \begin{cases} 1 & \lambda \in \mathcal{L}'_{n-1} \\ 0 & \lambda \notin \mathcal{L}'_{n-1} \end{cases} \approx p(\lambda)$$

that introduces the errors in both left-hand and right-hand sides of the equation (21).

From the expression (9) one see that as $C(\omega) \to 0$ we have $r_n \to r_{n-1}$ (implying $\mathcal{L}'_n \to \mathcal{L}'_{n-1}$) so the condition of the minimum norm:

$$\sum_{\omega \in \mathcal{W}'} C(\omega) C^*(\omega) = min \tag{22}$$

is a natural one for the regularization of the concerned problem.

Simultaneous determination of coefficients $C(\omega)$ from Eq. (21) with the condition (22) leads to time consuming algorithms. However, since the right side of the equation (21) is dominated by the diagonal terms $\Pi(0)$ ie. the normalized surface below $p(\lambda)$, one can establish the Gauss-Seidel type processes where in each iteration:

$$C(\omega) = -\frac{Z(\omega)}{\Pi(0)}.$$

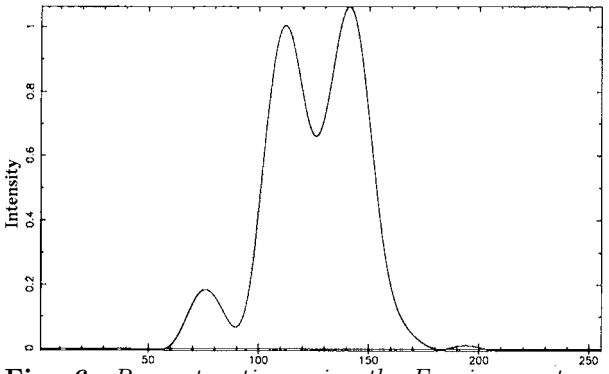

Fig. 6. *Reconstruction using the Fourier spectrum correction approach.*

The concrete iterative algorithm of that type we applied to our test problem resulted in reconstruction shown in Fig. 6.

In this example we used as the initial approximation the Fourier components of the non-constrained distribution:

$$\hat{R}_0(\omega) = \begin{cases} I(\omega)/S(\omega) & |\omega| \le \Omega_F \\ 0 & |\omega| > \Omega_F \end{cases}$$

where Ω_F is the Frieden's optimum processing bandwidth.

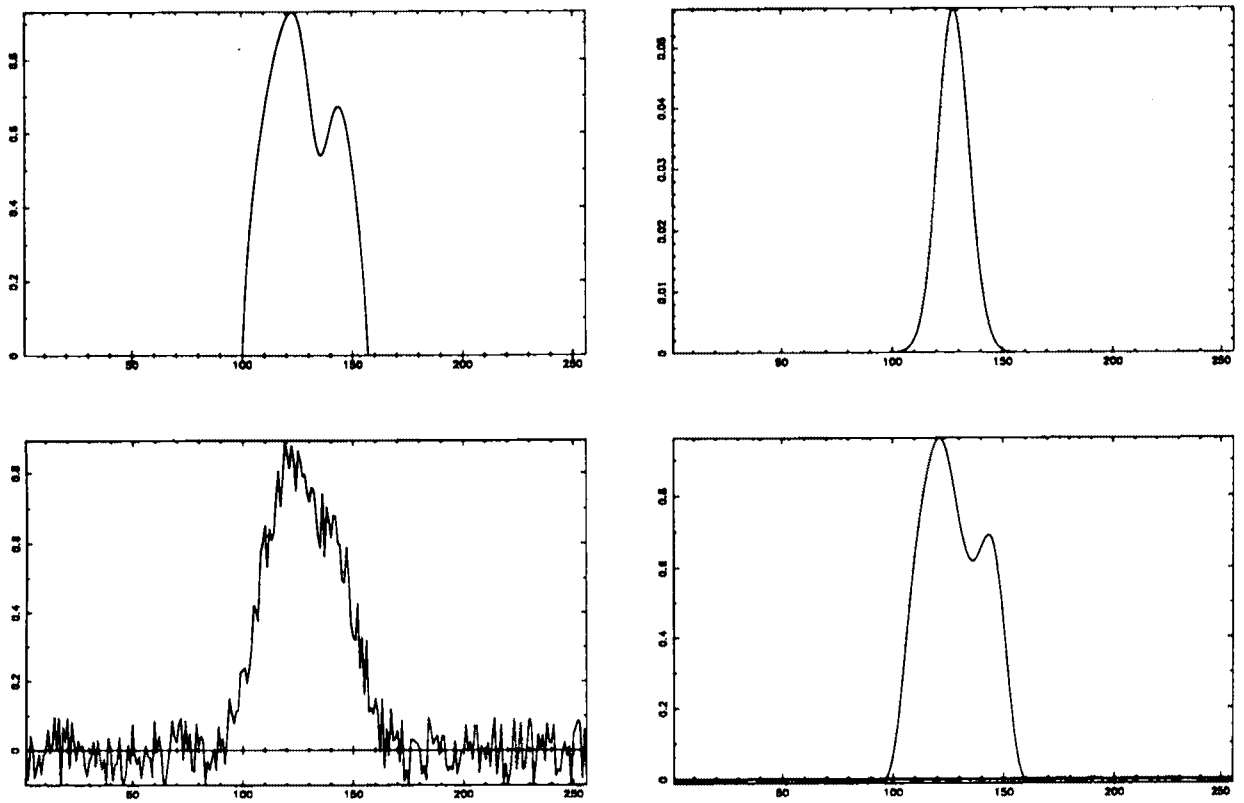

Fig. 7. *Test reconstruction using the Fourier spectrum correction approach with the prior knowledge of the finite extent of the rotation profile. The "observed" profile (bottom left) and the response function (top right) are used to reconstruct the original profile (top left). The resulting reconstruction (bottom right) shows (as compared with "observed" and original distribution) significant improvement in spectral resolution.*

This example shows that the constraint of positivity applied to the correction of the Fourier components of the signal leads to significant gains in resolution and reliability of the reconstructed distribution.

In order to examine the applicability of the constraint of finite extent we use the distorted stellar rotation profile (Jankov, 1992) as the object spectrum (Fig. 7. top left). Supposing known the "observed" profile (Fig. 7. bottom left) and the response function (Fig. 7. top right) the result of reconstruction is shown in Fig. 7. (bottom right). This example shows that the prior knowledge of the finite extent of rotation profile can be used to improve the resolution and signal-to-noise ratio of the rotationally broadened stellar spectra.

5. CONCLUSIONS

The approach based on correction of the Fourier components of the signal provides a way to produce efficient and fast algorithms to solve the deconvolution problem. It is consistent with the presence of noise in the data and has an advantage of high noise insensitivity.

The method gives the constrained estimate closest to the exact solution in the Euclidean data space allowing a big amount of *a priori* information that can be implemented.

The injection of general *a priori* information as positivity, smoothness, finite extent etc. allows to produce solutions that have good physical sense and leads to significant gains in resolution and reliability of the reconstructed distribution.

The method is general and can be easily implemented in a number of practical problems particularly in stellar spectroscopy.

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УСЛОВЉЕНА ДЕКОНВОЛУЦИЈА ЕКСПЛИЦИТНОМ КОРЕКЦИЈОМ ФУРИЈЕОВИХ КОМПОНЕНАТА НЕПОЗНАТЕ РАСПОДЕЛЕ

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Физичка ограничења су примењена у проблему деконволуције, експлицитном корекцијом шумом оптерећеним Фуријеових компонената сигнала. Метод даје условљену оцену оптималну у квадратном смислу тј. оцену најближу тачном решењу у Еуклидском простору решења. Особине метода су теоретски испитане и дате су одређене практичне примене у астрономској спектроскопији. Метод је упоређен са сличним процедурама екстраполације Фуријеовог спектра; као последица примена метода се препоручује нарочито у случају мерења са ниским односом сигнал-шум.