

MATHEMATICAL MODEL FOR THE 0.5 BILLION YEARS AGED SUN

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SUMMARY: An algorithm is given for constructing evolutionary tracks for a star with the mass equal to one solar mass. The presented model can be applied to the stars belonging to the inferior main sequence, which have the proton-proton reaction as energy source and present a radiative core and a convective shell. This paper presents an original way of solving the system of equations corresponding to the radiative nucleus by using Taylor's series in close vicinity to the center of the Sun. It also presents the numerical integration and the results for a 0.5 billion years aged solar model.

1. BASIC FORMULAE FOR THE EVOLUTIVE MODEL

Consider that for the radiative core of the Sun are valid the equations of hydrostatic equilibrium, mass distribution, luminosity and temperature (see, e.g., Menzel and others, 1963; Aller and McLaughlin, 1965; Cox and Giuli, 1968), given respectively by:

$$dP(r)/dr = -GM(r)\rho(r)/r^2, \quad (1)$$

$$dM(r)/dr = 4\pi r^2 \rho(r), \quad (2)$$

$$dL(r)/dr = 4\pi r^2 \rho(r)\varepsilon(r), \quad (3)$$

$$dT(r)/dr = \\ = -(3/(4ac))(\kappa(r)\rho(r)/T^3(r)) L(r)/(4\pi r^2), \quad (4)$$

where $P(r)$, $M(r)$, $L(r)$ and $T(r)$ represent the values of the pressure, the mass, the luminosity and the temperature in a point placed at the distance r from the center of the star. By using Schwarzschild's (1958) transformations:

$$\begin{aligned} P(r) &= pGM^2/(4\pi R^4), \\ M(r) &= qM, \\ L(r) &= fL, \\ T(r) &= t(\mu H/k)GM/R, \\ r &= x \cdot R, \end{aligned} \quad (5)$$

the dimensionless variables p , q , f , t , x are introduced. With these variables, the system (1) - (4) becomes:

$$\begin{aligned} dp/dx &= -pq/(tx^2), \\ dq/dx &= px^2/t, \\ df/dx &= C px^2/t, \\ dt/dx &= -D pf/(t^4x^2), \end{aligned} \quad (6)$$

where we have denoted:

$$\begin{aligned} C &= (M/L)(\varepsilon_{pp} + \varepsilon_{CN}), \\ D &= (3Lk^4/(64\pi^2 ac\mu^4 H^4 G^4 M^4))\kappa. \\ A &= (3Lk^4)/(64\pi^2 acH^4 G^4 M^4). \end{aligned} \quad (7)$$

The production of energy per gram-mass and per second due to the proton-proton reaction is given by the relation:

$$\begin{aligned} \varepsilon_{pp} = & \varepsilon_0 \rho X^2 (1 + 0.25 \rho^{1/2} T_6^{-3/2}) (1 + 0.012 T_6^{1/3} + \\ & 0.008 T_6^{2/3} + 0.00065 T_6) \cdot 10^6 T_6^{-2/3} \\ & \exp(-33.804 T_6^{-1/3}), \end{aligned} \quad (8)$$

where $\varepsilon_0 = 2.625$, ρ is the matter density expressed in g/cm^3 , T_6 is the temperature expressed in 10^6 K, while X denotes the hydrogen abundance.

Since the production of energy due to the carbon-nitrogen cycle will also be considered, this one is given by the relation:

$$\begin{aligned} \varepsilon_{CN} = & 7.94 \cdot 10^{27} (1 + 1.75 \rho^{1/2} T_6^{-3/2}) (1 + 0.027 T_6^{1/3} \\ & - 0.0037 T_6^{2/3} - 0.00007 T_6) \rho \cdot 0.00445 X^2 \\ & \exp(-152.313 T_6^{-1/3}) T_6^{-2/3}. \end{aligned} \quad (9)$$

The formulae (8) and (9) are given in Aller and McLaughlin (1965). The total production of energy per gram-mass and per second is given by the formula:

$$\varepsilon = \varepsilon_{pp} + \varepsilon_{CN}. \quad (10)$$

The opacity of stellar matter is considered to be given by the following analytical expressions (Hasselgrove and Hoyle, 1959):

– for $0.1 \leq T_6 < 10$:

$$\kappa = 0.19(1 + X) + a_1 \rho(1 + X); \quad (11)$$

– for $10 \leq T_6 < 20$:

$$\begin{aligned} \kappa = & 0.19(1 + X) + a_1 \rho(1 + X) + \\ & \rho(1 - 0.1 T_6)((1 + X)a_1 - b_1); \end{aligned} \quad (12)$$

– for $T_6 > 20$:

$$\kappa = 0.19(1 + X) + b_1 \rho; \quad (13)$$

where a_1 and b_1 have respectively the expressions:

$$\begin{aligned} a_1 = & 6.5 \cdot 10^4 (Z(1 - 0.05 T_6)/(T_6^2 + 2.5 T_6^4)) \\ & \exp(-7.75 \rho(1 + X)/T_6^3 + 4.15 \cdot 10^4 (X/(250 T_6^4 \\ & - T_6^2) + ((1 - X - Z)/(250 T_6^4)) (1 + 5.5 \\ & \exp(-|2/3 - 2.873 T_6|))) \exp(-2 \rho(1 + X)/T_6^3), \end{aligned} \quad (14)$$

and:

$$b_1 = (35 + 75X + 190Z)/T_6^{3.5}, \quad (15)$$

in which Z features the metals.

For the convective shell of the Sun, the equations (1) and (2), and the adiabatic equation:

$$P(r) = K \rho^\gamma(r), \quad \gamma = 5/3, \quad (16)$$

are considered to be valid. The ideal gas law is considered to be valid for the whole star.

Introducing the parameter:

$$(n + 1)_{rad} = d \log P / d \log T(r) = (1/D) q t / (p f), \quad (17)$$

the radiative equilibrium is ensured by the condition:

$$(n + 1)_{rad} > 2.5. \quad (18)$$

The system (1) - (4) is to be integrated with the boundary conditions (at center):

$$x = 0, \quad f = 0, \quad q = 0, \quad t = t_c, \quad p = p_c, \quad (19)$$

where t_c and p_c denote the dimensionless values of the temperature and pressure, respectively, at the Sun's center.

2. NUMERICAL SOLUTION OF THE MODEL

For the evolutive model which will be presented, we shall use the numerical results obtained by the author (Tatomir, 1986), which provide: pressure (P), temperature (T), dimensionless mass (q) and dimensionless luminosity (f) values in the points of a division:

$$\begin{aligned} x_1 < x_2 < \dots < x_{155}, \\ x_1 &= 0, \\ x_i &= (i - 1)h, \end{aligned} \quad (20)$$

where the integration step $h = 0.0058$ was taken.

The following values (corresponding to the parameters of the homogeneous model and the constants appearing in calculations) are used:

$$\begin{aligned} L_0 &= 3.12E + 33 \text{ (erg/s)}, \quad c = 2.9978E + 10 \text{ (cm/s)}, \\ R_0 &= 6.96E + 10 \text{ (cm)}, \\ G &= 6.672E - 08 \text{ (cm}^3\text{/(gs))}, \\ M_0 &= 1.99E + 33 \text{ (g)}, \quad Q_{pp}^* = 6.3E + 18 \text{ (erg/s)}, \\ k &= 1.379E - 16 \text{ (erg)}, \quad Q_{CN}^* = 6.0E + 18 \text{ (erg/s)}, \\ H &= 1.672E - 24 \text{ (g)}, \quad X = 0.709, \\ a &= 7.55E - 15 \text{ (dyn/cm}^2\text{)}, \quad Z = 0.021. \end{aligned} \quad (21)$$

The system (6) has a singularity in $x_1 = 0$; for calculating the values of p , q , f , t in the points x_1, x_2, \dots, x_7 , we have used in (14) the expanding in series method, obtaining:

$$\begin{aligned} p(x) = & p_0 - (1/6)(p_0^2/t_0^2)x^2 + ((1/45)p_0^3/t_0^4 - \\ & (DC/45)p_0^4/t_0^8)x^4 + 0 \cdot x^5 + \dots, \end{aligned} \quad (22)$$

$$\begin{aligned} q(x) = & (1/3)(p_0/t_0)x^3 + ((DC/30)p_0^3/t_0^7 - \\ & (1/30)p_0^2/t_0^3)x^5 + 0 \cdot x^6 + \dots, \end{aligned} \quad (23)$$

$$f(x) = (C/3)(p_0/t_0)x^3 + C((DC/30)p_0^3/t_0^7 - (1/30)p_0^2/t_0^3)x^5 + 0 \cdot x^6 + \dots, \quad (24)$$

$$t(x) = t_0 - (DC/6)(p_0^2/t_0^5)x^3 + ((DC/45)p_0^3/t_0^7 - (23D^2C^2/360)p_0^4/t_0^{11})x^4 + \dots, \quad (25)$$

where we have denoted $p_0 = p_c$ and $t_0 = t_c$.

By means of the values $X(x_i, 0)$, $\rho(x_i, 0)$, $T(x_i, 0)$ from the homogeneous model, one calculates the production of energy $\varepsilon_{pp}(x_i, 0) + \varepsilon_{CN}(x_i, 0)$ and the opacity $\kappa(x_i, 0)$, using the expressions (8)-(13).

As time step, we have chosen $\tau = 0.1 \cdot 10^9$ years; with this, the variation of chemical composition with time due to the nuclear reactions is given by:

$$X(x_i, \tau) = X(x_i, 0) - (\varepsilon_{pp}(x_i, 0)/Q_{pp}^* + \varepsilon_{CN}(x_i, 0)/Q_{CN}^*)\tau. \quad (26)$$

First it is calculated for each integration point x_i and for the epoch τ the molecular weight μ and the values of the coefficients C , D :

$$\mu(x_i, \tau) = 4/(3 + 5X(x_i, \tau) - Z), \quad (27)$$

$$C(x_i, \tau) = (1.99/3.12)(\varepsilon_{pp}(x_i, 0) + \varepsilon_{CN}(x_i, 0)), \quad (28)$$

$$D(x_i, \tau) = A\kappa(x_i, 0)/\mu^4(x_i, \tau), \quad (29)$$

where A is a numerical constant known from (7). In order to obtain the central values of density (ρ) and temperature (T) at the instant τ , the non-linear system:

$$\begin{aligned} \varepsilon_{pp}(0, 0) + \varepsilon_{CN}(0, 0) &= \varepsilon_{pp}(\rho, T, X(0, \tau)) + \\ \varepsilon_{CN}(\rho, T, X(0, \tau)), & \quad (30) \\ \kappa(0, 0) &= \kappa(\rho, T, X(0, \tau)) \end{aligned}$$

is solved by means of the Newton-Kantorovici method.

Next we will show how the Newton-Kantorovici method is used for the evolutive solar model.

Introducing the following notations:

$$f(\rho, T) = \varepsilon_{pp}(\rho, T, X(0, \tau)) + \varepsilon_{CN}(\rho, T, X(0, \tau)) - \varepsilon_{pp}(0, 0) - \varepsilon_{CN}(0, 0) \quad (31)$$

and

$$g(\rho, T) = \kappa(\rho, T, X(0, \tau)) - \kappa(0, 0) \quad (32)$$

the system (30) assumes the form :

$$\begin{aligned} f(\rho, T) &= 0 \\ g(\rho, T) &= 0 \end{aligned} \quad (33)$$

And now

$$H : R^2 \rightarrow R^2, \quad H(\rho, T) = \begin{bmatrix} f(\rho, T) \\ g(\rho, T) \end{bmatrix} \quad (34)$$

where f and g are some functions of the density ρ and the temperature T . Then the system (33) can be written under the form:

$$H(\rho, T) = 0 \quad (35)$$

To obtain the solution of the system (35) we start with an initial value of $\begin{bmatrix} \rho^k \\ T^k \end{bmatrix}$ which, in our case, is just the central value from the homogeneous model, $\begin{bmatrix} \rho^c \\ T^c \end{bmatrix}$. A better approximation of the solution of the system (33) can be obtained from the following formula:

$$\begin{aligned} \begin{bmatrix} \rho^{k+1} \\ T^{k+1} \end{bmatrix} &= \begin{bmatrix} \rho^k \\ T^k \end{bmatrix} - \\ & \begin{bmatrix} \partial f(\rho^k, T^k)/\partial \rho & \partial f(\rho^k, T^k)/\partial T \\ \partial g(\rho^k, T^k)/\partial \rho & \partial g(\rho^k, T^k)/\partial T \end{bmatrix}^{-1} \cdot \\ & \begin{bmatrix} f(\rho^k, T^k) \\ g(\rho^k, T^k) \end{bmatrix} \end{aligned} \quad (36)$$

We denote ρ_c^1 and T_c^1 the central values for the model of the type τ ; they are ρ^{k+1} and T^{k+1} in the formula (36). Starting from ρ_c^1 and T_c^1 we obtain tc1 and pc1 from Schwarzschild's transformations (5) and from the law of gases

$$P(r) = (1/\mu) \cdot (k/H) \cdot \rho(r) \cdot T(r) \quad (37)$$

With $t_0 = t_c^1$ and $p_0 = p_c^1$ and by means help of the series (22)-(25), we obtain the values for p , q , f , t in six points near the origin: x_2, x_3, \dots, x_7 . The system (6) is integrated using the Adams-Boshforth method of the sixth order (Moszynski, 1973)

$$\begin{aligned} V_{k+1} &= V_k + h((4277/1440)f_k - (7923/1440)f_{k-1} + \\ & (9982/1440)f_{k-2} - (7298/1440)f_{k-3} + \\ & (2877/1440)f_{k-4} - (475/1440)f_{k-5}) \end{aligned} \quad (38)$$

To improve the numerical results which have been obtained by way of the formula (38), the Adams-Moulton corrector method of the sixth order is used (Moszynski, 1973):

$$\begin{aligned} V_k &= V_{k-1} + h(475f_k + 1427f_{k-1} - 798f_{k-2} + \\ & 482f_{k-3} - 173f_{k-4} + 27f_{k-5})/1440 \end{aligned} \quad (39)$$

In this way we obtain the values for $t^1(x_i, \tau)$, $p^1(x_i, \tau)$, $f^1(x_i, \tau)$, $q^1(x_i, \tau)$, $\rho^1(x_i, \tau)$, $T^1(x_i, \tau)$. The integration of the system (6) proceeds as long as $(n+1)_{rad} \leq 2.5$. The integration of the model at the moment τ is repeated iteratively and we consider that for the iteration n we have the following values:

$$\begin{aligned} X^n(x_i, \tau), \rho^n(x_i, \tau), T^n(x_i, \tau), p^n(x_i, \tau), \\ f^n(x_i, \tau), q^n(x_i, \tau) \end{aligned} \quad (40)$$

The passing from the iteration n to the iteration $n + 1$ of the model at the moment τ is effected in this way:

$$X^{n+1}(x_i, \tau) = X(x_i, 0) - (1/2)((\varepsilon_{pp}(x_i, 0) + \varepsilon_{pp}^n(x_i, \tau))/Q_{pp}^* + (\varepsilon_{CN}(x_i, 0) + \varepsilon_{CN}^n(x_i, \tau))/Q_{CN}^*)\tau \quad (41)$$

$$\mu_{n+1}(x_i, \tau) = 4/(3 + 5X^n(x_i, \tau) - Z) \quad (42)$$

$$C^{n+1}(x_i, \tau) = (1.9891/3.826)(\varepsilon_{pp}^n(x_i, \tau) + \varepsilon_{CN}^n(x_i, \tau)) \quad (43)$$

$$D^{n+1}(x_i, \tau) = A \cdot \kappa^n(x_i, \tau)/(\mu^n(x_i, \tau))^4 \quad (44)$$

The central values of the model ρ_c^{n+1} , T_c^{n+1} at the moment τ and at the iteration $n + 1$ are obtained from the system:

$$\begin{aligned} & (1/2)(\varepsilon_{pp}(0, 0) + \varepsilon_{pp}^n(0, \tau) + \varepsilon_{CN}(0, 0) + \varepsilon_{CN}^n(0, \tau)) \\ & = \varepsilon_{pp}(\rho, T, X^{n+1}(0, \tau)) + \varepsilon_{CN}(\rho, T, X^{n+1}(0, \tau)) \\ & (1/2)(\kappa(0, 0) + \kappa^n(0, \tau)) = \kappa(\rho, T, X^{n+1}(0, \tau)) \end{aligned} \quad (45)$$

The conditions to stop the iterations of the models at the moment τ are:

$$|\rho_c^n - \rho_c^{n+1}| < \varepsilon_1 \quad (46)$$

$$|T_c^n - T_c^{n+1}| < \varepsilon_2 \quad (47)$$

When the conditions (46) and (47) are fulfilled, it is considered that the model from the iteration n is good and this model will be considered as being the one at the moment τ .

The passing from a model at the moment $m \cdot \tau$ to a model at the moment $(m + 1) \cdot \tau$ is done in the identical way as the passing from the model at the moment $\tau = 0$ to the moment at the moment τ .

3. NUMERICAL RESULTS

Table 1 lists the numerical values featuring the solar model which corresponds to the age $\tau = 0.5 \cdot 10^9$ years.

The quantities appearing in Table 1 are:

- P – pressure (expressed in 10^{18} dyn/cm²);
- T – temperature (expressed in 10^6 K);
- ρ – density (expressed in g/cm³);
- X – hydrogen abundance;
- x – non-dimensional radius;
- q – non-dimensional mass;
- f – non-dimensional luminosity.

Table 1.

x	P	q	f	T	ρ	X
0.0	0.18156	0.0	0.0	14.146	96.7660	0.6902
0.023	0.17833	0.0008	0.0078	14.055	95.6119	0.6909
0.052	0.16552	0.0093	0.0796	13.701	90.8560	0.6935
0.081	0.14499	0.0332	0.2462	13.105	83.9744	0.6971
0.104	0.12536	0.0666	0.4233	12.500	75.0414	0.7001
0.203	0.04778	0.3324	0.9698	9.384	37.8816	0.7076
0.301	0.01209	0.6381	1.0538	6.538	13.7513	0.7089
0.400	0.00249	0.8357	1.0577	4.459	4.1509	0.7089
0.505	0.00041	0.9334	1.0578	2.955	1.0466	0.7090
0.603	0.69E-04	0.9682	1.0578	2.017	0.2556	0.7090
0.701	0.10E-04	0.9794	1.0578	1.401	0.5418E-01	0.7090
0.800	0.12E-05	0.9824	1.0578	0.965	0.9701E-02	0.7090
0.893	0.13E-06	0.9830	1.0578	0.693	0.1418E-02	0.7090

Table 2.

τ	f	L	T_{ef}	$\log(L/L_0)$	$\log T_{\text{ef}}$
0.5	1.0578	3.300	5569.23	-0.072	3.745

In order to compare the numerical data of the models obtained by the author to the observational data, we proposed to plot the model onto the $(\log(L/L_0), \log T_{ef})$ -plane. In this respect we can represent our models on the observational diagram Hertzsprung-Russell.

Since the energy is not produced in the region of the convective shell, namely $\varepsilon = 0$ and $L = \text{constant}$, the luminosity of the models is considered as being given by:

$$L = L_0 f(x_r) = L_0 f(0.893) \quad (48)$$

The effective temperature is obtained from the well-known relationship valid for the black body:

$$L = 4\pi R^2 \sigma_R T_{ef}^4 \quad (49)$$

where $R = 6.96 \cdot 10^{10}$ cm, while σ_R denotes the constant of Stefan-Boltzmann: $\sigma_R = 5.6687 \cdot 10^{-5}$ erg·cm⁻²·s⁻¹·deg⁻⁴.

Table 2 provides the values of the luminosity and effective temperature corresponding to the models obtained by the author.

In this Table, the time τ is expressed in 10^9 years, f is (as previously) the non-dimensional luminosity, the luminosity L is expressed in 10^{33} erg/s, while the effective temperature T_{ef} is expressed in K.

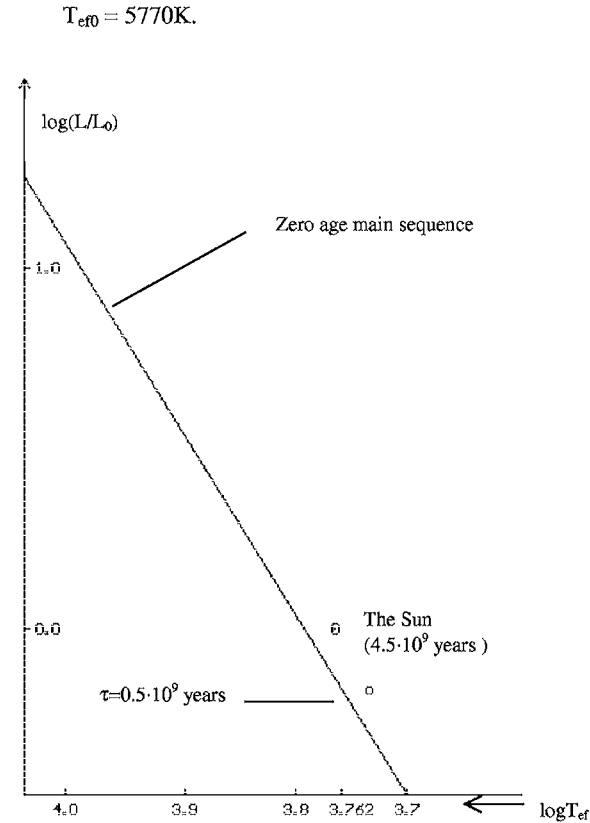


Fig. 1.

In Fig. 1 there are plotted onto the Hertzsprung-Russell diagram the model obtained by the author. There are evolutionary models of stars of $1M_0$ belonging to Population I, corresponding to the epochs (ages) $\tau = 0.5 \cdot 10^9$ years. In comparison, the present-day Sun (aged about $4.5 \cdot 10^9$ years) is plotted on the diagram too, using the following data:

$$\begin{aligned} R_0 &= 6.96 \cdot 10^{10} \text{ cm,} \\ L_0 &= 3.826 \cdot 10^{33} \text{ erg/s,} \\ T_{ef0} &= 5770 \text{ K.} \end{aligned} \quad (50)$$

Since there aren't any other calculated models of the chemical composition on the condition of $0.4 \cdot 10^9$ age, considered in this paper, nor any by other authors, the only thing that remains is to compare the model to the observational data.

The position of the $4.5 \cdot 10^9$ aged Sun on the H-R diagram is presented in Fig. 1, but the intermediate positions at different times can not be deduced in an observational way. The evolution of the Sun along the main sequence is not linear because the inner temperature increases in time, but the hydrogen abundance X decreases, modifying the values of the energy production and of the opacity given by the formulas (8)-(15). The position of the $0.4 \cdot 10^9$ aged model and of the actual Sun in the parallel positions with the main sequence and the distance between them, show that the model which I have calculated approximates very well the position that the Sun would have occupied in the H-R diagram if its age were $0.4 \cdot 10^9$ years.

The papers quoted in the text were consulted at writing this paper. Other papers quoted in the references are recommended to be read for a better understanding of the studied theme.

The system (6), which has to be integrated on the condition (19), presents an indeterminacy under the form of $0/0$. For the elimination of the indeterminacy we applied the series (22)-(25).

We have shown how the Newton-Kantorovicz method can be used for the evolutive solar models and we have integrated the system (6) using the method of successive approximations. The fact that the model, which has been calculated, gives good results in the comparison to the observation, entails that this model placed in the H-R diagram occupies a correct position.

In conclusion, the original way of numerical solving, which is presented in Chapter 3 can be used by all the evolutive models which have a radiative nucleus and a convective cover.

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МАТЕМАТИЧКИ МОДЕЛ ЗА СУНЦЕ СТАРО 0.5 МИЛИЈАРДИ ГОДИНА

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Дат је алгоритам за конструкцију еволуционог процеса за звезде чија је маса једнака Сунчевој маси. Приказани модел може се применити на звезде које припадају доњем главном низу, а које за извор енергије имају протон-протон реакцију и имају радиативно језгро и конвективни омотач. Овим радом приказан

је оригинални начин решавања система једначина које одговарају радиативном језгру коришћењем Тејлоревих редова у непосредној близини средишта Сунца. Он такође приказује нумеричку интеграцију и резултате за Сунчев модел старости 0.5 милијарди година.