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R3 FLUIDS

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SUMMARY: The current paper is aimed at getting more insight on three main points concerning large-scale astrophysical systems, namely: (i) formulation of tensor virial equations from the standpoint of analytical mechanics; (ii) investigation on the role of systematic and random motions with respect to virial equilibrium configurations; (iii) determination of extent to which systematic and random motions are equivalent in flattening or elongating the shape of a mass distribution. The tensor virial equations are formulated regardless of the nature of the system and its constituents, by generalizing and extending a procedure used for the scalar virial equations in presence of discrete subunits (Landau and Lifchitz 1966). In particular, the self potential-energy tensor is shown to be symmetric with respect to the exchange of the indices, $(E_{\mathrm{pot}})_{pq} = (E_{\mathrm{pot}})_{qp}$. Then the results are extended to continuous mass distributions. The role of systematic and random motions in collisionless, ideal, self-gravitating fluids is analysed in detail including radial and tangential velocity dispersion on the equatorial plane, and the related mean angular velocity, $\overline{\Omega}$, is conceived as a figure rotation. R3 fluids are defined as ideal, self-gravitating fluids in virial equilibrium, with systematic rotation around a principal axis of inertia, taken to be x_3. The related virial equations are written in terms of the moment of inertia tensor, I_{pq}, the self potential-energy tensor, $(E_{\mathrm{pot}})_{pq}$, and the generalized anisotropy tensor, ζ_{pq} (Caimmi and Marmo 2005, Caimmi 2006a). Additional effort is devoted to the investigation of the properties of axisymmetric and triaxial configurations. A unified theory of systematic and random motions is developed for R3 fluids, taking into consideration imaginary rotation (Caimmi 1996b, 2006a), and a number of theorems previously stated for homeoidally striated Jacobi ellipsoids (Caimmi 2006a) are extended to the more general case of R3 fluids. The effect of random motion excess is shown to be equivalent to an additional real or imaginary rotation, respectively, inducing flattening (along the equatorial plane) or elongation (along the rotation axis). Then it is realized that a R3 fluid always admits an adjoint configuration with isotropic random velocity distribution. In addition, further constraints are established on the amount of random velocity anisotropy along the principal axes, for triaxial configurations. A necessary condition is formulated for the occurrence of bifurcation points from axisymmetric to triaxial configurations in virial equilibrium, which is independent of the anisotropy parameters. A particularization of general relations is made to the special case of homeoidally striated Jacobi ellipsoid, and some previously known results (Caimmi 2006a) are reproduced.

Key words. Galaxies: clusters: general – Galaxies: haloes

1. INTRODUCTION

Large-scale celestial objects, such as stellar systems, galaxy clusters, and (non baryonic) dark matter haloes predicted by current QCDM cosmologies, may safely be represented as collisionless, ideal self-gravitating fluids. The related flow equation takes the same formal expression as in their collisional counterpart, with the exception that the pressure force is generalized in terms of a stress tensor, allowing different rms velocities along different directions [e.g. Binney and Tremaine 1987 (hereafter quoted as BT87), Chap. 4, § 2]. Accordingly, collisionless fluids can be flattened equally well by rotation (with respect to a selected axis) and/or anisotropic random velocity distribution i.e. anisotropic pressure (e.g. Caimmi 2006a, hereafter quoted as C06[1]). In fact, giant elliptical galaxies exhibit a negligible amount of (systematic) rotation, and their shape is mainly due to anisotropic pressure (e.g. Bertola and Capaccioli 1975, Illingworth 1977, 1981, Schechter and Gunn 1979, BT87, Chap. 4, § 36).

If the dark energy (quintessence, if it is due to a scalar field rolling down its own potential) clusters on scales of dark matter haloes (e.g. Mota and van de Bruck 2004, Maor and Lahav 2005, Nunes and Mota 2006), quintessence components within 2-component systems (matter + quintessence) may safely be represented as collisionless, ideal self-interacting fluids where both the nature and the strength of the interaction are expected, in general, to be different from the case of gravitation.

Collisionless fluids of astrophysical and cosmological interest range over about ten orders of magnitude in mass, say from globular clusters to galaxy clusters, provided gaseous i.e. collisional component may safely be neglected. Therefore, it seems necessary to investigate the role of systematic and random motions in making virialized collisionless fluids. To this respect, the virial theorem in tensor form may be a useful tool. According to the standard procedure, the tensor virial equations are determined along the following steps (e.g. Binney 1978, 2005, Wiegandt 1982a,b, BT87, Chap. 4, § 3): (a) start with the collisionless Boltzmann equations; (b) derive a set of moment equations; (c) integrate the above set of moment equations, under some simplifying assunptions.

To the (limited) knowledge of the author, no attempt can be found in the literature where (i) the tensor virial equations are formulated from the standpoint of analytical mechanics, (ii) the role of systematic and random motions is clearly stated, and (iii) the equivalence between systematic and random motions in flattening or elongating the boundary, is clearly established.

Concerning (i), the tensor virial equations could be determined regardless of the nature of the system and its constituents, by generalizing and extending a procedure used for the scalar virial equations [Landau and Lifchitz 1966 (hereafter quoted as LL66), Chap. II, § 10]. Regarding (ii), preliminary considerations reported in previous attempts [Caimmi 1996a,b, Caimmi and Marmo 2005 (hereafter quoted as CM05), C06] should be further improved and developed. In dealing with (iii), the definition of imaginary rotation allows an interpretation of rms velocity excess in terms of systematic rotation around a fixed principal axis (Caimmi 1996b, C06, Caimmi 2006b,[1] which could be inserted in the context under discussion. A detailed investigation on the above mentioned points is the purpose of the present attempt.

To this aim, the simplest case of one-component systems shall be considered. To tell the truth, two-component (or more) systems are common between large-scale astrophysical objects (e.g. gas + stars; baryonic + dark matter; matter + dark energy), and further study should be devoted to the above mentioned topics. For instance, both the formulation of the virial theorem (e.g. Limber 1959, Brosche et al. 1983, Caimmi and Secco 1992, Horellou and Berge 2005, Maor and Lahav 2005, Percival 2005, Caimmi 2007), and the occurrence of a bifurcation point from axisymmetric to triaxial configurations (e.g. Durisen 1978, Pacheco et al. 1986, Caimmi 1996a, Balaguera-Antolinez et al. 2006) are influenced by the presence of a secondary subsystem.

The current paper is organized as follows. A general formulation of the tensor virial theorem which holds, in particular, for the gravitational interaction is provided in Section 2. The properties of R3 fluids, defined as ideal, self-gravitating fluids in virial equilibrium, rotating around a principal axis, taken to be x_3, are studied in Section 3, where a number of theorems previously stated for homeoidally striated Jacobi ellipsoids (C06) are extended to the more general case of R3 fluids. A unified theory of systematic and random motions is provided in Section 4, where some general relations are particularized to the special case of homeoidally striated Jacobi ellipsoids, and previously known results (Caimmi 1996a,b, C06) are reproduced. Some concluding remarks are drawn in Section 5, and a few arguments are treated in more detail in the Appendix.

2. THE TENSOR VIRIAL THEOREM

A general procedure used for the formulation of the scalar virial theorem (LL66, Chap. II, § 10) will be followed here in the derivation of the tensor virial theorem. Let us take into consideration a mechanical system made of N particles, referred to an inertial reference frame. Let $(x_i)_r$, $(v_i)_r$, be the position and velocity components related to i-th particle, and m_i the mass, $1 \leq i \leq N$, $1 \leq r \leq 3$.

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[1] A more extended file including an earlier version of the above quoted paper is available at the arxiv electronic site, as astro-ph/0507314.

The kinetic-energy tensor:

$$(E_{\rm kin})_{pq}=\frac12\sum_{i=1}^N m_i(v_i)_p(v_i)_q\quad;\tag{1}$$

is a function of N or $2N$ variables, $(v_i)_r$, $1\le i\le N$, $1\le p\le 3$, $1\le q\le 3$, for selected p and q, depending on whether the tensor components are diagonal or non diagonal, respectively. The kinetic-energy tensor is manifestly symmetric with respect to the exchange of the indices:

$$(E_{\rm kin})_{pq}=(E_{\rm kin})_{qp}\quad;\tag{2}$$

and the trace is the kinetic energy:

$$E_{\rm kin}=\sum_{s=1}^3(E_{\rm kin})_{ss}=\frac12\sum_{i=1}^N\sum_{s=1}^3 m_i(v_i)_s^2\quad;\tag{3}$$

which is a function of $3N$ variables, $(v_i)_r$, $1\le i\le N$, $1\le r\le 3$. The first partial derivatives are:

$$(p_i)_r=\frac{\partial E_{\rm kin}}{\partial(v_i)_r}=m_i(v_i)_r\quad;\tag{4}$$

where $(p_i)_r$, $1\le i\le N$, $1\le r\le 3$, is the momentum component of i particle (e.g. LL66, Chap. II, § 7).

The combination of Eqs. (1) and (4) yields:

$$2(E_{\rm kin})_{pq}=\sum_{i=1}^N(v_i)_p(p_i)_q\quad;\tag{5}$$

which is equivalent to:

$$2(E_{\rm kin})_{pq}=\frac{d}{dt}\left[\sum_{i=1}^N(x_i)_p(p_i)_q\right]-\sum_{i=1}^N(x_i)_p(\dot p_i)_q\quad;\tag{6}$$

or, using Newton's equations (e.g. LL66, Cap. I, § 5):

$$2(E_{\rm kin})_{pq}=\frac{d}{dt}\left[\sum_{i=1}^N(x_i)_p(p_i)_q\right]+\sum_{i=1}^N(x_i)_p\frac{\partial E_{\rm pot}}{\partial(x_i)_q}\quad;\tag{7}$$

where $E_{\rm pot}[(x_i)_r]$, $1\le i\le N$, $1\le r\le 3$, is the self potential energy.

The last term on the right-hand side of Eq. (7) defines a tensor, the trace of which is usually named the virial of the system (Clausius 1870). In the author's opinion, it would be better to refer to the virial and its parent tensor as the virial potential energy and the virial potential-energy tensor, respectively.

If the self potential energy is a homogeneous function of the coordinates, of degree χ, then the following relation holds:

$$E_{\rm pot}[\zeta\times(x_i)_r]=\zeta^\chi E_{\rm pot}[(x_i)_r]\quad;\tag{8}$$

which, in turn, implies:

$$(E_{\rm pot})_{pq}[\zeta\times(x_i)_r]=\zeta^\chi(E_{\rm pot})_{pq}[(x_i)_r]\quad;\tag{9}$$

where $(E_{\rm pot})_{pq}$ is defined as the self potential-energy tensor:

$$(E_{\rm pot})_{pq}=\frac1\chi\sum_{i=1}^N(x_i)_p\frac{\partial E_{\rm pot}}{\partial(x_i)_q}\quad;\tag{10}$$

and ζ is a positive real number, provided the algebraic product, $\zeta\times(x_i)_r$, $1\le i\le N$, $1\le r\le 3$, belongs to the domain of $E_{\rm pot}$. The special case, $\chi=-1$, corresponds to the gravitational interaction.

With regard to the self potential energy for the system under consideration, the Euler theorem reads:

$$\sum_{s=1}^3\sum_{i=1}^N(x_i)_s\frac{\partial E_{\rm pot}}{\partial(x_i)_s}=\chi E_{\rm pot}\quad;\tag{11}$$

and the combination of Eqs. (10) and (11) yields:

$$\sum_{s=1}^3(E_{\rm pot})_{ss}=E_{\rm pot}\quad;\tag{12}$$

as expected.

The substitution of Eq. (10) into (7) yields:

$$\frac{d}{dt}\left[\sum_{i=1}^N(x_i)_p(p_i)_q\right]=2(E_{\rm kin})_{pq}-\chi(E_{\rm pot})_{pq}\quad;\tag{13}$$

and the sum of Eq. (13) and its counterpart with the indices p and q interchanged, reads:

$$\frac{d}{dt}\sum_{i=1}^N[(x_i)_p(p_i)_q+(x_i)_q(p_i)_p]=2[(E_{\rm kin})_{pq}+(E_{\rm kin})_{qp}]-\chi[(E_{\rm pot})_{pq}+(E_{\rm pot})_{qp}]\quad,\tag{14}$$

Let us define the moment of inertia tensor[2] [e.g. Chandrasekhar 1969 (hereafter quoted as C69), Chap. 2, § 9; BT87, Chap. 4, § 3]:

$$I_{pq}=\sum_{i=1}^N m_i(x_i)_p(x_i)_q\quad;\tag{15a}$$

$$\sum_{s=1}^3 I_{ss}=I\quad;\tag{15b}$$

where I is the total moment of inertia of the system, with respect to the centre of mass. Owing to Eq. (4), the first time derivative is:

$$\dot I_{pq}=\frac{dI_{pq}}{dt}=\sum_{i=1}^N[(x_i)_p(p_i)_q+(x_i)_q(p_i)_p]\quad;\tag{16}$$

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[2] In this formulation, the moment of inertia with respect to a coordinate axis, x_r, is $I_r=I_{pp}+I_{qq}$, $r\ne p\ne q$. For a different formulation where $I_r=I_{rr}$, $r=1,2,3$, see LL66 (Chap. VI, § 32).

and the combination of Eqs. (2), (14), and (16), yields:

$$\ddot{I}_{pq} = 4(E_{\rm kin})_{pq} - \chi[(E_{\rm pot})_{pq} + (E_{\rm pot})_{qp}] \quad ; \qquad (17)$$

on the other hand, the difference of Eq. (13) with its counterpart where the indices, p and q, are interchanged, reads:

$$\frac{d}{dt}\sum_{i=1}^{N}[(x_i)_p(p_i)_q - (x_i)_q(p_i)_p] = -\chi[(E_{\rm pot})_{pq} - (E_{\rm pot})_{qp}] \quad . \qquad (18)$$

owing to Eq. (2).

With regard to the vectors, $\vec{r}_i[(x_i)_1, (x_i)_2, (x_i)_3]$ and $\vec{p}_i[(p_i)_1, (p_i)_2, (p_i)_3]$, and to the vector product, $\vec{J}_i = \vec{r}_i \times \vec{p}_i$, the sum on the left-hand side of Eq. (18) reads (e.g. Spiegel 1968, Chap. 2.2, §§ 11-12):

$$\sum_{i=1}^{N}[(x_i)_p(p_i)_q - (x_i)_q(p_i)_p] = \sum_{i=1}^{N}\overrightarrow{vers}(x_r)\cdot(\vec{r}_i \times \vec{p}_i) = \overrightarrow{vers}(x_r)\cdot\sum_{i=1}^{N}\vec{J}_i = \overrightarrow{vers}(x_r)\cdot\vec{J} = J_r \quad ; \qquad (19)$$

where $\overrightarrow{vers}(x_r)$ is the versor, or unit vector, parallel to the coordinate axis, x_r, $r \neq p \neq q$, and J is the total angular momentum of the system.

The combination of Eqs. (18) and (19) yields:

$$\frac{dJ_r}{dt} = -\chi[(E_{\rm pot})_{pq} - (E_{\rm pot})_{qp}] \quad ; \qquad (20)$$

and the conservation of angular momentum, which always holds for isolated systems (e.g. LL66, Chap. 2, § 9), implies the symmetry of the self potential-energy tensor with respect to the exchange of the indices:

$$(E_{\rm pot})_{pq} = (E_{\rm pot})_{qp} \quad ; \qquad (21)$$

so that Eq. (17) takes the form:

$$\frac{1}{2}\ddot{I}_{pq} = 2(E_{\rm kin})_{pq} - \chi(E_{\rm pot})_{pq} \quad , \qquad (22)$$

which makes the virial equations of the second order (for the special case of gravitational interaction, $\chi = -1$, see e.g. C69, Chap. 2, § 11; BT87, Chap. 4, § 3).

The further constraint:

$$\ddot{I}_{pq} = 0 \quad ; \qquad 1 \le p \le 3 \quad ; \qquad 1 \le q \le 3 \quad ; \qquad (23)$$

makes Eqs. (22) reduce to:

$$2(E_{\rm kin})_{pq} - \chi(E_{\rm pot})_{pq} = 0 \ ; \quad 1 \le p \le 3 \ ; \quad 1 \le q \le 3 \ ; \qquad (24)$$

which is the expression of the virial theorem in tensor form[3]. Strictly speaking, it holds when the moment of inertia tensor has a linear dependence on time, $I_{pq} = k_{pq}t + c_{pq}$, where k_{pq} and c_{pq} are constants. The special case, $k_{pq} = 0$, $1 \le p \le 3$, $1 \le q \le 3$, is related to dynamical or hydrostatic equilibrium (e.g. BT87, Chap. 4, § 3).

An alternative constraint is that the first time derivatives of the moment of inertia tensor are bounded, as:

$$|\dot{I}_{pq}(t)| \le M_{pq} \ ; \quad 1 \le p \le 3 \ ; \quad 1 \le q \le 3 \ ; \qquad (25)$$

where M_{pq} are suitable real numbers. Accordingly, it can be seen that the time average of the second time derivatives of the moment of inertia tensor are null (e.g. LL66, Chap. II, § 10):

$$\overline{\ddot{I}_{pq}} = 0 \quad ; \qquad 1 \le p \le 3 \quad ; \qquad 1 \le q \le 3 \quad ; \qquad (26)$$

which makes Eq. (22) reduce to:

$$2\overline{(E_{\rm kin})_{pq}} - \chi\overline{(E_{\rm pot})_{pq}} = 0 \ ; \quad 1 \le p \le 3 \ ; \quad 1 \le q \le 3 \ ; \qquad (27)$$

where time averages are calculated over a sufficiently long (ideally infinite) step (e.g. LL66, Chap. 2, § 10). In presence of periodic motions (e.g. a homogeneous sphere undergoing coherent oscillations), time averages can be calculated over a single (or a multiple) period.

For the sake of simplicity, in what follows the tensor virial theorem will be expressed by Eq. (24) where the kinetic-energy and self potential-energy tensors are to be considered as instantaneous or time averaged, depending on whether the constraint defined by Eq. (23) or (26) holds.

The particularization of Eq. (24) to diagonal components, after summation on both sides, produces:

$$2E_{\rm kin} - \chi E_{\rm pot} = 0 \quad ; \qquad (28)$$

which is the expression of the virial theorem in scalar form (e.g. LL66, Chap. II, § 10). Special cases are (a) Newtonian and Coulombian interaction, $\chi = -1$, and (b) Hookeian interaction, $\chi = 2$. If the system is in dynamical or hydrostatic equilibrium, mean values coincide with instantaneous values.

The above results are quite general and hold regardless of the nature of the system and its constituents, provided no dissipation and/or external interaction occur. With regard to a specified system, the sole restrictions to be made are (i) the evolution takes place within a finite region of the phase hyperspace i.e. $0 \le |(x_i)_r| < M_{x_i}$, $0 \le |(v_i)_r| < M_{v_i}$, $1 \le i \le N$, $1 \le r \le 3$, where M_{x_i}, M_{v_i}, are convenient real numbers, and (ii) the self potential energy

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[3] Some authors prefer a more general formulation, expressed by Eq. (22) (e.g. BT87, Chap. 4, § 3). On the other hand, a more restricted formulation, expressed by Eq. (24), has a closer connection with the scalar virial theorem, which explains the choice adopted here.

is a homogeneous function of the $3N$ coordinates, of degree χ.

In dealing with continuous matter distributions rather then with mass points, the particle mass, m_i, has to be replaced by the mass within an infinitesimal volume element, $dm = \rho(x_1, x_2, x_3)\,dx_1\,dx_2\,dx_3$, where ρ is the density, and an integration has to be performed over the whole volume, S, instead of a summation over the coordinates related to all the particles. For further details see e.g. Limber (1959). Accordingly, the kinetic-energy tensor and the kinetic energy attain their usual expressions (C69, Chap. 2, § 9):

$$(E_{\rm kin})_{pq} = \frac{1}{2}\int_S \rho(x_1, x_2, x_3) v_p v_q \, d^3S \quad ; \tag{29}$$

$$(E_{\rm kin}) = \frac{1}{2}\int_S \rho(x_1, x_2, x_3) \sum_{s=1}^{3} v_s^2 \, d^3S \quad ; \tag{30}$$

on the other hand, the self potential-energy tensor and the self potential energy read:

$$(E_{\rm pot})_{pq} = -\frac{1}{\chi}\int_S \rho_\chi(x_1, x_2, x_3) x_p \frac{\partial \mathcal{V}}{\partial x_q} \, d^3S \ ; \tag{31}$$

$$(E_{\rm pot}) = -\frac{1}{\chi}\int_S \rho_\chi(x_1, x_2, x_3) \sum_{s=1}^{3} x_s \frac{\partial \mathcal{V}}{\partial x_s} \, d^3S \ , \tag{32}$$

where ρ_χ is a charge density and $\mathcal{V}$ is a potential function, defined as the tidal potential energy due to the whole charge distribution, related to the point under consideration, with the unit charge placed therein. For further details, see Appendix A.

Let us define the total-energy tensor, as:

$$E_{pq} = (E_{\rm kin})_{pq} + (E_{\rm pot})_{pq} \quad ; \tag{33}$$

owing to Eqs. (3) and (12), the related trace:

$$E = \sum_{s=1}^{3} E_{ss} = E_{\rm kin} + E_{\rm pot} \tag{34}$$

is the total energy.

The combination of Eqs. (24), (33), and (28), (34), respectively, yields:

$$(E_{\rm kin})_{pq} = \frac{\chi}{\chi+2} E_{pq} \quad , \tag{35}$$

$$(E_{\rm pot})_{pq} = \frac{2}{\chi+2} E_{pq} \quad , \tag{36}$$

for tensor components, and:

$$E_{\rm kin} = \frac{\chi}{\chi+2} E \quad , \tag{37}$$

$$E_{\rm pot} = \frac{2}{\chi+2} E \quad , \tag{38}$$

for tensor traces.

3. SYSTEMATIC AND RANDOM MOTIONS

3.1. Basic ideas

Let a collisionless, self-gravitating fluid be referred to an inertial frame, $(\mathsf{O}x_1x_2x_3)$, where (without loss of generality) the origin coincides with the centre of mass. The number of particles within an infinitesimal hypervolume of the phase hyperspace at the time, t, is:

$$d^6\mathcal{N} = f(x_1, x_2, x_3, v_1, v_2, v_3, t) \times dx_1\,dx_2\,dx_3\,dv_1\,dv_2\,dv_3 \quad ; \tag{39}$$

where $f \geq 0$ is the distribution function. The number of particles within an infinitesimal volume of the ordinary space at the time, t, is:

$$d^3\mathcal{N} = dx_1\,dx_2\,dx_3 \times \iiint f(x_1, x_2, x_3, v_1, v_2, v_3, t)\,dv_1\,dv_2\,dv_3; \tag{40}$$

where the integration has to be performed over the whole volume in velocity space. The number density related to the infinitesimal volume element, $d^3S = dx_1\,dx_2\,dx_3$, at the time, t, is:

$$n(x_1, x_2, x_3, t) = \frac{d^3\mathcal{N}}{d^3S} = \iiint f(x_1, x_2, x_3, v_1, v_2, v_3, t)\,dv_1\,dv_2\,dv_3; \tag{41}$$

if, in addition, the total particle number, $\mathcal{N}$, and the total mass, M, are conserved, then the following normalization conditions hold:

$$\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int f(x_1, x_2, x_3, v_1, v_2, v_3, t)\,dx_1\,dx_2\,dx_3 \times dv_1\,dv_2\,dv_3 = \mathcal{N} \quad ; \tag{42}$$

$$\iiint \rho(x_1, x_2, x_3, t)\,dx_1\,dx_2\,dx_3 = M \quad ; \tag{43}$$

where ρ is the mass density of the infinitesimal volume element, d^3S, and the integrations have to be carried over the whole hypervolume in phase hyperspace and the whole volume in ordinary space, respectively.

From the physical point of view, the volume element is finite instead of infinitesimal, but still containing a large amount of particles which, on the other hand, is negligible with respect to the total number. Accordingly, the following relations hold:

$$1 << \Delta\mathcal{N}(x_1, x_2, x_3, t) << \mathcal{N} \quad ; \tag{44a}$$

$$\max(m_i) << \Delta M(x_1, x_2, x_3, t) << M \quad ; \tag{44b}$$

where $\Delta\mathcal{N}$ and ΔM represent the particle total number and total mass within the volume element, ΔS, at the time, t, and m_i is the mass of i-th particle, $1 \le i \le \Delta\mathcal{N}$. The related total mass, ΔM, may be expressed as:

$$\Delta M(x_1, x_2, x_3, t) = \sum_{i=1}^{\Delta\mathcal{N}} m_i \quad ; \tag{45}$$

and the mean particle mass within the volume element, ΔS, at the time, t, reads:

$$\overline{m}(x_1, x_2, x_3, t) = \frac{\Delta M(x_1, x_2, x_3, t)}{\Delta\mathcal{N}(x_1, x_2, x_3, t)} \quad ; \tag{46}$$

according to the general definition of arithmetic mean.

From the standpoint of a continuous mass distribution, the following changes have to be made: $\Delta S \to d^3S$; $\Delta\mathcal{N}(x_1, x_2, x_3, t) \to d^3\mathcal{N}$; $\Delta M(x_1, x_2, x_3, t) \to \rho(x_1, x_2, x_3, t)\, d^3S$; and Eq. (46) takes the form:

$$\begin{aligned}\overline{m}(x_1, x_2, x_3, t) &= \rho(x_1, x_2, x_3, t)\frac{d^3S}{d^3\mathcal{N}} \\ &= \frac{\rho(x_1, x_2, x_3, t)}{n(x_1, x_2, x_3, t)} \quad ;\end{aligned} \tag{47}$$

in terms of mass density and number density.

With regard to Eqs. (42) and (43), equivalent expressions are:

$$\int\!\!\int\!\!\int n(x_1, x_2, x_3, t)\, dx_1\, dx_2\, dx_3 = \mathcal{N} \quad ; \tag{48}$$

$$\begin{aligned}&\int\!\!\int\!\!\int \rho(x_1, x_2, x_3, t)\, dx_1\, dx_2\, dx_3 = \\ &\int\!\!\int\!\!\int \overline{m}(x_1, x_2, x_3, t) n(x_1, x_2, x_3, t)\, dx_1\, dx_2\, dx_3 \\ &= M \quad ;\end{aligned} \tag{49}$$

and the division of both sides of Eq. (49) by their counterparts in Eq. (48), yields:

$$\overline{m} = \frac{M}{\mathcal{N}} \quad , \tag{50}$$

where, owing to the theorem of the mean, $\overline{m}$ is the particle mass averaged over the whole volume. Total mass and particle number conservation imply a time independent mean particle mass, $\overline{m}$. If, in addition, particles with different masses are uniformly distributed throughout the whole volume, then the mean particle mass within a generic volume element, d^3S, equals the mean particle mass within the boundary, as:

$$\overline{m}(x_1, x_2, x_3, t) = \overline{m} \quad ; \tag{51}$$

and the system may be considered, in any respect, as made of $\mathcal{N}$ identical particles of mass $\overline{m}$. Accordingly, Eq. (47) reads:

$$\rho(x_1, x_2, x_3, t) = \overline{m}\, n(x_1, x_2, x_3, t) \quad ; \tag{52}$$

which implies direct proportionality between mass density and number density.

Using again the theorem of the mean, let us define the mean velocity component, $\overline{v_p}$, and the mean product velocity component, $\overline{v_pv_q}$, within a generic infinitesimal volume element, $d^3S = dx_1\, dx_2\, dx_3$, at the time, t, as:

$$\begin{aligned}&\overline{v_p}(x_1, x_2, x_3, t) = \\ &\frac{\int\int\int f(x_1, x_2, x_3, v_1, v_2, v_3, t) v_p\, dv_1\, dv_2\, dv_3}{\int\int\int f(x_1, x_2, x_3, v_1, v_2, v_3, t)\, dv_1\, dv_2\, dv_3} \quad ;\end{aligned} \tag{53}$$

$$\begin{aligned}&\overline{v_pv_q}(x_1, x_2, x_3, t) = \\ &\frac{\int\int\int f(x_1, x_2, x_3, v_1, v_2, v_3, t) v_pv_q\, dv_1\, dv_2\, dv_3}{\int\int\int f(x_1, x_2, x_3, v_1, v_2, v_3, t)\, dv_1\, dv_2\, dv_3} ,\end{aligned} \tag{54}$$

or, using Eq. (41),

$$\begin{aligned}&\overline{v_p}(x_1, x_2, x_3, t) = \\ &\frac{\int\int\int f(x_1, x_2, x_3, v_1, v_2, v_3, t) v_p\, dv_1\, dv_2\, dv_3}{n(x_1, x_2, x_3, t)} \quad ;\end{aligned} \tag{55}$$

$$\begin{aligned}&\overline{v_pv_q}(x_1, x_2, x_3, t) = \\ &\frac{\int\int\int f(x_1, x_2, x_3, v_1, v_2, v_3, t) v_pv_q\, dv_1\, dv_2\, dv_3}{n(x_1, x_2, x_3, t)} ,\end{aligned} \tag{56}$$

in terms of the number density, n.

Let us define the distribution function in the velocity space:

$$F(x_1, x_2, x_3, v_1, v_2, v_3, t) = \frac{f(x_1, x_2, x_3, v_1, v_2, v_3, t)}{n(x_1, x_2, x_3, t)} , \tag{57}$$

which, owing to Eq. (41), satisfies the normalization condition:

$$\int\!\!\int\!\!\int F(x_1, x_2, x_3, v_1, v_2, v_3, t)\, dv_1\, dv_2\, dv_3 = 1 \quad , \tag{58}$$

and the substitution of Eq. (57) into (55) and (56) yields:

$$\begin{aligned}&\overline{v_p}(x_1, x_2, x_3, t) = \\ &\int\!\!\int\!\!\int F(x_1, x_2, x_3, v_1, v_2, v_3, t) v_p\, dv_1\, dv_2\, dv_3 \quad ;\end{aligned} \tag{59}$$

$$\begin{aligned}&\overline{v_pv_q}(x_1, x_2, x_3, t) = \\ &\int\!\!\int\!\!\int F(x_1, x_2, x_3, v_1, v_2, v_3, t) v_pv_q\, dv_1\, dv_2\, dv_3 ,\end{aligned} \tag{60}$$

in terms of the distribution function, F.

From a statistical standpoint, the distribution function, F, may be interpreted as a probability density in the velocity space, where $F(x_1, x_2, x_3, v_1, v_2, v_3, t)\, dv_1\, dv_2\, dv_3$ represents the probability of finding a particle inside the volume element, d^3S, at the time, t, with velocity components in the range, $v_r \mp dv_r/2$, $r = 1, 2, 3$. In this view, the velocity components, v_p, and the product of the velocity components, $v_p v_q$, may be considered as random variables. According to the general definition of variance and covariance (e.g. Oliva and Terrasi 1976, Chap. II, § 2.7), the following relations hold:

$$(v_p^2)^* = (v_p^*)^2 + \sigma_{v_p}^2 \quad ; \tag{61}$$

$$(v_p v_q)^* = v_p^* v_q^* + \sigma_{v_p v_q} \quad , \tag{62}$$

where the asterisk denotes the expectation value of the related distribution, σ_a^2 is the mathematical (intended as opposite to empirical) variance and σ_{ab} is the mathematical covariance. Accordingly, if two random variables, a and b, are independent, correlated, or anticorrelated, the mathematical covariance, σ_{ab}, is null, positive, or negative, respectively.

With regard to an infinitesimal volume element, d^3S, at the time, t, the first terms on the right-hand sides of Eqs. (61) and (62) are related to the velocity components of the centre of mass, while the second terms are related to velocity components with respect to the centre of mass.

Expectation values and mathematical variances and covariances are a priori quantities which cannot be determined by data collections. The related observables are arithmetic means and empirical variances and covariances (e.g. Oliva and Terrasi 1976, Chap. IV, § 4.3), and Eqs. (61) and (62) translate into:

$$\overline{(v_p^2)} = (\overline{v_p})^2 + \sigma_{v_p}^2 \quad ; \tag{63}$$

$$\overline{v_p v_q} = \overline{v_p}\ \overline{v_q} + \sigma_{v_p v_q} \quad , \tag{64}$$

where the notation of variances and covariances has been left unchanged, for sake of simplicity. Finally, Eq. (63) may also be conglobed into Eq. (64), using the definitions:

$$\overline{v_r v_r} = \overline{(v_r^2)} \quad ; \qquad \sigma_{v_r v_r} = \sigma_{v_r}^2 \quad , \tag{65}$$

where $\sigma_{v_p v_q}$ may be considered as the generic component of an empirical covariance tensor. If velocity components are independent, $\sigma_{v_p v_q} = \delta_{pq}\sigma_{v_p}^2$, where δ_{pq} is the Kronecker symbol, Eqs. (63) and (64) merge into:

$$\overline{v_p v_q} = \overline{v_p}\ \overline{v_q} + \delta_{pq}\sigma_{pp}^2 \quad ; \tag{66}$$

where $\sigma_{rr}^2 = \sigma_{v_r v_r} = \sigma_{v_r}^2$ to simplify the notation, and considering σ_{rr} as velocity dispersions related to random motions, with regard to a generic infinitesimal volume element, d^3S, at the time, t.

Having the centre of mass of the system been chosen as origin of the (inertial) reference frame, mean (over the whole volume) velocity components along coordinate axes are necessarily null: $\overline{v_r} = 0$; (e.g. LL66, Chap. II, § 8). Accordingly, $\sigma_{rr}^2 = \overline{(v_r^2)}$, $\sigma_{pq}^2 = 0$, $p \neq q$, and the random square velocity tensor, $\sigma_{pq}^2 = \overline{v_p v_q}$, is diagonal.

3.2. Radial and tangential velocity dispersion on the equatorial plane

Aiming at expressing the contribution of systematic and random motions along the equatorial plane in terms of angular velocities, a detailed analysis of radial and tangential velocity dispersion on the equatorial plane is needed.

Let us define the axes of the system as the distances between opposite points along the principal axes of inertia, belonging to the external surface, and semiaxes the distances between the tops of the axes and the centre of mass. In general, semiaxes on opposite sides are different in length. Let the coordinate axis x_3, containing the axis a_3, be chosen as rotation axis, without loss of generality, as the reference frame may arbitrarily be oriented, provided the origin coincides with the centre of mass. Let us define the principal plane, $(\mathsf{O}x_1x_2)$, as the equatorial plane of the system.

For a particle placed at a position $\vec{r} = (x_1, x_2, x_3)$, and moving at a velocity $\vec{v} = (v_1, v_2, v_3)$, let $\vec{r}_{\rm eq} = (x_1, x_2)$ and $\vec{v}_{\rm eq} = (v_1, v_2,)$ be the related projections onto the equatorial plane (see Fig. 1).

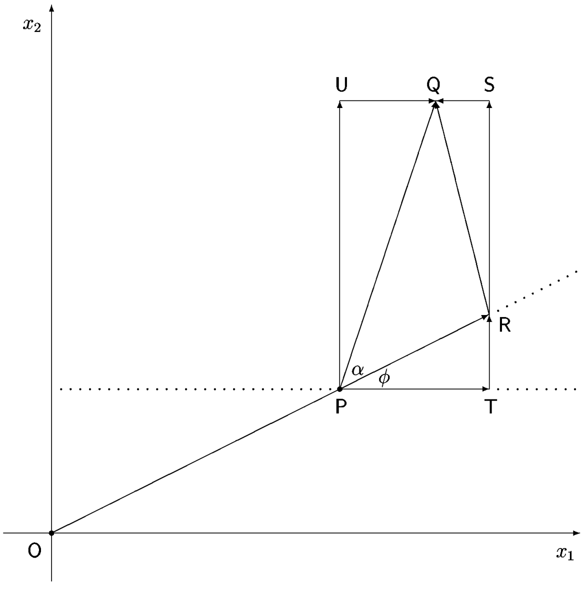

Figure 1:

$\overline{\mathsf{OP}} = r_{\rm eq} = (x_1^2 + x_2^2)^{1/2} = w$;

$\overline{\mathsf{RQ}} = v_\phi = v_{\rm eq} \sin\alpha$;

$\overline{\mathsf{SQ}} = (v_\phi)_1 = -v_\phi \sin\phi$;

$\overline{\mathsf{PT}} = (v_w)_1 = -v_w \cos\phi$;

$\phi = \widehat{\mathsf{TPR}}$; $\qquad \alpha = \widehat{\mathsf{RPQ}}$;

$\overline{\mathsf{UQ}} = v_1 = v_{\rm eq} \cos(\alpha + \phi)$;

$\overline{\mathsf{PQ}} = v_{\rm eq} = (v_1^2 + v_2^2)^{1/2}$;

$\overline{\mathsf{PR}} = v_w = v_{\rm eq} \cos\alpha$;

$\overline{\mathsf{RS}} = (v_\phi)_2 = v_\phi \cos\phi$;

$\overline{\mathsf{TR}} = (v_w)_2 = v_w \sin\phi$;

$\alpha + \phi = \widehat{\mathsf{PQU}}$;

$\overline{\mathsf{PU}} = v_2 = v_{\rm eq} \sin(\alpha + \phi)$;

Fig. 1. *Radial (v_w) and tangential (v_ϕ) velocity components on the equatorial plane, ($\mathsf{O}x_1x_2$). The projections onto the equatorial plane of the vector radius $\vec{r}$, and the velocity $\vec{v}$, are denoted by $r_{\rm eq}$ and $v_{\rm eq}$, respectively, with regard to a generic particle.*

Cartesian velocity components may be expressed as the algebraic sum of radial and tangential velocity projection on the related direction, as:

$$\begin{aligned} v_1 &= v_{\mathrm{eq}}\cos(\alpha+\phi) = (v_w)_1 + (v_\phi)_1 \\ &= v_w\cos\phi - v_\phi\sin\phi \quad ; \qquad (67\mathrm{a}) \\ v_2 &= v_{\mathrm{eq}}\sin(\alpha+\phi) = (v_w)_2 + (v_\phi)_2 \\ &= v_w\sin\phi + v_\phi\cos\phi \quad ; \qquad (67\mathrm{b}) \end{aligned}$$

conversely, radial and tangential velocity components may be expressed as the algebraic sum of Cartesian velocity projections on the related direction, as:

$$\begin{aligned} v_w &= v_{\mathrm{eq}}\cos\alpha = (v_1)_w + (v_2)_w \\ &= v_1\cos\phi + v_2\sin\phi \quad ; \qquad (68\mathrm{a}) \\ v_\phi &= v_{\mathrm{eq}}\sin\alpha = (v_1)_\phi + (v_2)_\phi \\ &= -v_1\sin\phi + v_2\cos\phi \quad ; \qquad (68\mathrm{b}) \end{aligned}$$

where $(v_\mu)_r = [\vec{v}\cdot\overrightarrow{vers}(\mu)]\,\overrightarrow{vers}(\mu)\cdot\overrightarrow{vers}(x_r)$; $(v_r)_\mu = [\vec{v}\cdot\overrightarrow{vers}(x_r)]\,\overrightarrow{vers}(x_r)\cdot\overrightarrow{vers}(\mu)$; $\mu = w,\phi$; $r = 1,2$; and $\overrightarrow{vers}(d)$ is the unit vector with positive orientation, along the d direction.

For a generic infinitesimal volume element, d^3S, at the time, t, radial and tangential velocity components, defined by Eqs. (68), may be considered as random variables. Owing to a theorem of statistics[4], the expectation values of the related distributions read:

$$\begin{aligned} v_w^* &= v_1^*\cos\phi + v_2^*\sin\phi \quad ; \qquad (69\mathrm{a}) \\ v_\phi^* &= -v_1^*\sin\phi + v_2^*\cos\phi \quad ; \qquad (69\mathrm{b}) \end{aligned}$$

similarly, the expectation values of the distributions depending on the random variables, v_w^2 and v_ϕ^2, are found to be:

$$\begin{aligned} &(v_w^2)^* = \\ &(v_1^2)^*\cos^2\phi + (v_2^2)^*\sin^2\phi + 2(v_1v_2)^*\cos\phi\sin\phi; (70\mathrm{a}) \\ &(v_\phi^2)^* = \\ &(v_1^2)^*\sin^2\phi + (v_2^2)^*\cos^2\phi - 2(v_1v_2)^*\sin\phi\cos\phi; (70\mathrm{b}) \end{aligned}$$

and, using the general definitions expressed by Eqs. (61) and (62), the related mathematical variances read:

$$\begin{aligned} &\sigma_{v_w}^2 = (v_w^2)^* - (v_w^*)^2 \\ &= \sigma_{v_1}^2\cos^2\phi + \sigma_{v_2}^2\sin^2\phi + 2\sigma_{v_1v_2}\cos\phi\sin\phi \quad ; (71\mathrm{a}) \\ &\sigma_{v_\phi}^2 = (v_\phi^2)^* - (v_\phi^*)^2 \\ &= \sigma_{v_1}^2\sin^2\phi + \sigma_{v_2}^2\cos^2\phi - 2\sigma_{v_1v_2}\sin\phi\cos\phi \quad ; (71\mathrm{b}) \end{aligned}$$

where the mathematical covariance, $\sigma_{v_1v_2}$, is null provided the related velocity components, v_1 and v_2, are independent. The validity of the relations:

$$\begin{aligned} (v_w^*)^2 + (v_\phi^*)^2 &= (v_1^*)^2 + (v_2^*)^2 \quad ; \qquad (72) \\ (v_w^2)^* + (v_\phi^2)^* &= (v_1^2)^* + (v_2^2)^* \quad ; \qquad (73) \\ \sigma_{v_w}^2 + \sigma_{v_\phi}^2 &= \sigma_{v_1}^2 + \sigma_{v_2}^2 \quad ; \qquad (74) \end{aligned}$$

can be easily checked.

In terms of the related observables, arithmetic means and empirical variances and covariances, Eqs. (69), (70), and (71) translate into:

$$\begin{aligned} \overline{v_w} &= \overline{v_1}\cos\phi + \overline{v_2}\sin\phi \quad ; \qquad (75\mathrm{a}) \\ \overline{v_\phi} &= -\overline{v_1}\sin\phi + \overline{v_2}\cos\phi \quad ; \qquad (75\mathrm{b}) \end{aligned}$$

$$\begin{aligned} &\overline{(v_w^2)} = \\ &\overline{(v_1^2)}\cos^2\phi + \overline{(v_2^2)}\sin^2\phi + 2\overline{v_1v_2}\cos\phi\sin\phi \quad ; \ (76\mathrm{a}) \\ &\overline{(v_\phi^2)} = \\ &\overline{(v_1^2)}\sin^2\phi + \overline{(v_2^2)}\cos^2\phi - 2\overline{v_1v_2}\sin\phi\cos\phi \quad ; \ (76\mathrm{b}) \end{aligned}$$

$$\begin{aligned} &\sigma_{v_w}^2 = \overline{(v_w^2)} - (\overline{v_w})^2 \\ &= \sigma_{v_1}^2\cos^2\phi + \sigma_{v_2}^2\sin^2\phi + 2\sigma_{v_1v_2}\cos\phi\sin\phi \quad ; (77\mathrm{a}) \\ &\sigma_{v_\phi}^2 = \overline{(v_\phi^2)} - (\overline{v_\phi})^2 \\ &= \sigma_{v_1}^2\sin^2\phi + \sigma_{v_2}^2\cos^2\phi - 2\sigma_{v_1v_2}\sin\phi\cos\phi \quad ; (77\mathrm{b}) \end{aligned}$$

where the notation of variances and covariances has been left unchanged, for the sake of simplicity. The validity of the relations:

$$\begin{aligned} (\overline{v_w})^2 + (\overline{v_\phi})^2 &= (\overline{v_1})^2 + (\overline{v_2})^2 \quad ; \qquad (78) \\ \overline{(v_w^2)} + \overline{(v_\phi^2)} &= \overline{(v_1^2)} + \overline{(v_2^2)} \quad ; \qquad (79) \\ \sigma_{v_w}^2 + \sigma_{v_\phi}^2 &= \sigma_{v_1}^2 + \sigma_{v_2}^2 \quad ; \qquad (80) \end{aligned}$$

can also be easily checked.

If velocity components are independent, $\sigma_{v_pv_q} = \delta_{pq}\sigma_{v_p}^2$, Eqs. (77) reduce to:

$$\begin{aligned} \sigma_{ww}^2 &= \sigma_{11}^2\cos^2\phi + \sigma_{22}^2\sin^2\phi \quad ; \qquad (81\mathrm{a}) \\ \sigma_{\phi\phi}^2 &= \sigma_{11}^2\sin^2\phi + \sigma_{22}^2\cos^2\phi \quad ; \qquad (81\mathrm{b}) \end{aligned}$$

where, as already mentioned (see Eq. (66)), $\sigma_{rr}^2 = \sigma_{v_rv_r} = \sigma_{v_r}^2$ to simplify the notation and considering σ_{rr} as velocity dispersions related to random motions with regard to a generic infinitesimal volume element, d^3S, at the time, t.

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[4] Let m_1, m_2, ..., m_n, be random variables and $f_1(m_1)\,dm_1$, $f_2(m_2)\,dm_2$, ..., $f_n(m_n)\,dm_n$, related distributions, $m = \sum_{k=1}^n \alpha_k m_k$ an additional random variable, where α_k are coefficients, and $f(m)\,dm$ a related distribution. Then the expectation value, m^*, can be expressed via the above linear combination of the expectation values, m_1^*, m_2^*, ..., m_n^*, as: $m^* = \sum_{k=1}^n \alpha_k m_k^*$.

For the whole volume, S, at the time, t, let us define positive and negative equatorial radial velocity components, v_w, as directed outwards and inwards, respectively, and positive and negative equatorial tangential velocity components, v_ϕ, as related to counterclockwise and clockwise motion, respectively, around the rotation axis.

Owing to the above mentioned theorem of statistics, the following relations hold for for expectation values and mathematical variances related to the distributions depending on radial and tangential velocity components on the equatorial plane:

$$v_w^* = \frac{1}{M}\int\int\int [v_w(x_1,x_2,x_3,t)]^*\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{82a}$$

$$v_\phi^* = \frac{1}{M}\int\int\int [v_\phi(x_1,x_2,x_3,t)]^*\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{82b}$$

$$(v_w^2)^* = \frac{1}{M}\int\int\int [v_w^2(x_1,x_2,x_3,t)]^*\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{83a}$$

$$(v_\phi^2)^* = \frac{1}{M}\int\int\int [v_\phi^2(x_1,x_2,x_3,t)]^*\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{83b}$$

$$\sigma_{v_w}^2 = (v_w^2)^* - (v_w^*)^2 \quad ; \tag{84a}$$

$$\sigma_{v_\phi}^2 = (v_\phi^2)^* - (v_\phi^*)^2 \quad ; \tag{84b}$$

where the validity of Eqs. (72), (73), and (74) remains unchanged.

In terms of the related observables, arithmetic means and empirical variances, Eqs. (82), (83), and (84), translate into:

$$\overline{v_w} = \frac{1}{M}\int\int\int \overline{v_w}(x_1,x_2,x_3,t)\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{85a}$$

$$\overline{v_\phi} = \frac{1}{M}\int\int\int \overline{v_\phi}(x_1,x_2,x_3,t)\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{85b}$$

$$\overline{v_w^2} = \frac{1}{M}\int\int\int \overline{v_w^2}(x_1,x_2,x_3,t)\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{86a}$$

$$\overline{v_\phi^2} = \frac{1}{M}\int\int\int \overline{v_\phi^2}(x_1,x_2,x_3,t)]\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{86b}$$

$$\sigma_{v_w}^2 = \sigma_{ww}^2 = \overline{(v_w^2)} - (\overline{v_w})^2 \quad ; \tag{87a}$$

$$\sigma_{v_\phi}^2 = \sigma_{\phi\phi}^2 = \overline{(v_\phi^2)} - (\overline{v_\phi})^2 \quad ; \tag{87b}$$

where the validity of Eqs. (78), (79), and (80) remains unchanged.

With regard to the angular velocity, Ω, and the related moment of inertia, I_3, the counterparts of Eqs. (85b) and (86b) read:

$$\overline{\Omega} = \frac{1}{I_3}\int\int\int \overline{\Omega}(x_1,x_2,x_3,t)w\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{88}$$

$$\overline{\Omega^2} = \frac{1}{I_3}\int\int\int \overline{\Omega^2}(x_1,x_2,x_3,t)w^2\rho(x_1,x_2,x_3,t) \times dx_1\,dx_2\,dx_3 \quad ; \tag{89}$$

$$\sigma_\Omega^2 = \overline{\Omega^2} - \overline{\Omega}^2 \quad ; \tag{90}$$

$$I_3 = \int\int\int w^2\rho(x_1,x_2,x_3,t)\,dx_1\,dx_2\,dx_3 \quad ; \tag{91}$$

where the angular velocity, $\overline{\Omega}$, may be conceived as a figure rotation, in the sense that the mean is null when performed in a reference frame in rigid rotation at the same rate. To this respect, particles with different masses must be uniformly distributed within the region of phase hyperspace accessible to the system.

Owing to Eqs. (84b), (89), and (90), the following relation holds:

$$M(\overline{v_\phi}^2 + \sigma_{\phi\phi}^2) = M\overline{v_\phi^2} = I_3\overline{\Omega^2} = I_3(\overline{\Omega}^2 + \sigma_\Omega^2) \quad ; \tag{92}$$

where the mean angular velocity, $\overline{\Omega}$, and the empirical variance, σ_Ω^2, are related to systematic and random motions, respectively, around the rotation axis. Accordingly, Eq. (92) may be splitted into:

$$M\overline{v_\phi}^2 = I_3\overline{\Omega}^2 \quad ; \tag{93a}$$

$$M\sigma_{\phi\phi}^2 = I_3\sigma_\Omega^2 \quad ; \tag{93b}$$

which express the contribution of systematic and random motions along the equatorial plane, in terms of tangential and angular velocities.

The mean radial velocity component, $\overline{v_w}$, is related to the motion of the centre of mass along the equatorial plane. On the other hand, the centre of mass coincides with the origin of the coordinates, which implies $\overline{v_w} = 0$ and, in turn, $\overline{(v_w^2)} = \sigma_{ww}^2$.

The mean tangential velocity component, $\overline{v_\phi}$, and the empirical variance, $\sigma_{\phi\phi}^2$, are related to systematic and random motions, respectively, around the rotation axis. The diagonal components of the kinetic-energy tensor may be expressed in terms of the above mentioned contributions, as:

$$T_{kk} = (T_{\rm sys})_{kk} + (T_{\rm rdm})_{kk} \quad ; \tag{94}$$

where $k = w, \phi$ in the case under discussion, and:

$$(T_{\rm sys})_{ww} = 0 \quad ; \tag{95a}$$

$$(T_{\rm rdm})_{ww} = \frac{1}{2} M \sigma_{ww}^2 \quad ; \tag{95b}$$

$$(T_{\rm sys})_{\phi\phi} = \frac{1}{2} I_3 \overline{\Omega}^2 \quad ; \tag{96a}$$

$$(T_{\rm rdm})_{\phi\phi} = \frac{1}{2} I_3 \sigma_\Omega^2 \quad ; \tag{96b}$$

where the indices, *sys* and *rdm*, denote systematic and random motions, respectively.

The global contribution:

$$T_{\phi\phi} = \frac{1}{2} I_3 \overline{\Omega^2} \quad ; \tag{97}$$

depends only on the mass distribution, via the moment of inertia, I_3, related to the rotation axis, x_3, and the tangential velocity component on the equatorial plane, via the rms angular velocity, $\overline{\Omega^2}$, regardless of the amount of systematic and random motions along the direction under discussion (e.g. Meza 2002, C06).

The contribution of the kinetic-energy tensor component, $T_{\phi\phi}$, to the kinetic-energy tensor components, T_{11} and T_{22}, owing to Eqs. (96) and (97), is:

$$(T_{\phi\phi})_{qq} = \frac{1}{2} I_{qq} \overline{\Omega^2} \quad ; \qquad q = 1, 2 \quad ; \tag{98}$$

$$[(T_{\rm sys})_{\phi\phi}]_{qq} = \frac{1}{2} I_{qq} \overline{\Omega}^2 \quad ; \qquad q = 1, 2 \quad ; \tag{99}$$

$$[(T_{\rm rdm})_{\phi\phi}]_{qq} = \frac{1}{2} I_{qq} (\overline{\Omega^2} - \overline{\Omega}^2) \quad ; \qquad q = 1, 2 \quad ; \tag{100}$$

$$I_{pq} = \int\int\int x_p x_q \rho(x_1, x_2, x_3, t)\, dx_1\, dx_2\, dx_3 \quad ; \tag{101}$$

$$I_3 = I_{11} + I_{22} \quad ; \tag{102}$$

where I_{pq} is the moment of inertia tensor.

With regard to equatorial radial kinetic-energy tensor components, the combination of Eqs. (80), (90), and (93) yields:

$$\sigma_{ww}^2 = \sigma_{11}^2 + \sigma_{22}^2 - \frac{I_3}{M} (\overline{\Omega^2} - \overline{\Omega}^2) \quad ; \tag{103}$$

and the contribution of the kinetic-energy tensor component T_{ww} to the kinetic-energy tensor components T_{11} and T_{22} owing to Eqs. (94) and (102), is:

$$(T_{ww})_{qq} = \frac{1}{2} M \sigma_{qq}^2 - \frac{1}{2} I_{qq} (\overline{\Omega^2} - \overline{\Omega}^2) \quad ; \qquad q = 1, 2 \quad ; \tag{104}$$

to be used together with Eq. (98).

It is worth noticing that anisotropic random velocity components distributions, $\sigma_{TT} \neq \sigma_{RR}$, are not necessarily related to the shape of the system, while $\sigma_{WW} \neq 2\sigma_{33}$ is (e.g. BT87, Chap. 4, § 3), where T, R, and W, denote tangential, radial, and equatorial velocity components, respectively. In fact, a spherically symmetric mass distribution could, in principle, allow purely radial or circular orbits.

3.3. Virial equilibrium configurations

The particularization of Eq. (22) to Newtonian interaction, $\chi = -1$, after combination with Eqs. (98) and (104), allows the formulation of the virial equations for the case under discussion. The result is:

$$\frac{1}{2} \ddot{I}_{qq} = I_{qq} \overline{\Omega}^2 + M \sigma_{qq}^2 + (E_{\rm pot})_{qq} \quad ; \quad q = 1, 2 \quad ; \tag{105a}$$

$$\frac{1}{2} \ddot{I}_{33} = M \sigma_{33}^2 + (E_{\rm pot})_{33} \quad ; \tag{105b}$$

where $\overline{v_3} = 0$, the system centre of mass having been chosen as origin of the reference frame, and the tensors are diagonal provided the coordinate axes coincide with the principal axes of inertia.

The virial equations of the second order, expressed by Eq. (22), in particular Eqs. (105), imply the validity of the following assumptions.

(i) The mechanical system under consideration (in particular a collisionless, self-gravitating fluid) is isolated (e.g. LL66, Chap. I, § 5), which implies angular momentum conservation (e.g. LL66, Chap. II, § 9).

(ii) The potential energy is a homogeneous function of the coordinates with the degree χ (in particular, $\chi = -1$).

The validity of the further assumption, either (iii-a) The generic component of the moment of inertia tensor depends linearly on time, according to Eq. (23); or, alternatively:
(iii-b) The first time derivative of the generic component of the moment of inertia tensor is a bounded function, according to Eq. (25); makes Eqs. (105) reduce to:

$$I_{qq} \overline{\Omega}^2 + M \sigma_{qq}^2 + (E_{\rm pot})_{qq} = 0; \quad q = 1, 2; \tag{106a}$$

$$M \sigma_{33}^2 + (E_{\rm pot})_{33} = 0 \quad ; \tag{106b}$$

where the variables are to be regarded as instantaneous or averaged over a sufficiently long time, depending of whether assumption (iii-a) or (iii-b), respectively, has been chosen.

A more general formulation of Eqs. (106), which includes instantaneuos configurations with assumption (iii-b), is:

$$I_{qq} \overline{\Omega}^2 + M \zeta_{qq} \sigma^2 + (E_{\rm pot})_{qq} = 0 \quad ; \quad q = 1, 2 \quad ; \tag{107a}$$

$$M \zeta_{33} \sigma^2 + (E_{\rm pot})_{33} = 0 \quad ; \tag{107b}$$

$$\sigma^2 = \sigma_{11}^2 + \sigma_{22}^2 + \sigma_{33}^2 \quad ; \tag{107c}$$

$$\zeta_{rr}=\frac{(\tilde{E}_{\rm rdm})_{rr}}{E_{\rm rdm}}=\frac{\tilde{\sigma}_{rr}^2}{\sigma^2}\quad;\qquad r=1,2,3\quad;\qquad(107\text{d})$$

$$\zeta_{11}+\zeta_{22}+\zeta_{33}=\frac{\tilde{E}_{\rm rdm}}{E_{\rm rdm}}=\frac{\tilde{\sigma}^2}{\sigma^2}=\zeta\quad;\qquad(107\text{e})$$

where ζ_{rr} may be understood as anisotropy parameters (CM05, C06), $E_{\rm rdm}$ is the random kinetic energy, and $\tilde{E}_{\rm rdm}$ is the effective random kinetic energy i.e. the right amount needed for an instantaneous configuration to satisfy Eqs. (106). For further details refer to Appendix B. Generalized anisotropy parameters smaller or larger than $\zeta/3$ imply, respectively, lack or excess of random motions along the related direction. On the other hand, the ratios:

$$\tilde{\zeta}_{rr}=\frac{(\tilde{E}_{\rm rdm})_{rr}}{\tilde{E}_{\rm rdm}}=\frac{\zeta_{rr}}{\zeta}\quad;\qquad r=1,2,3\quad;\qquad(108\text{a})$$

$$\tilde{\zeta}_{11}+\tilde{\zeta}_{22}+\tilde{\zeta}_{33}=1\quad;\qquad(108\text{b})$$

may be understood as effective anisotropy parameters (CM05, C06).

The parameter ζ may be conceived as a virial index, where $\zeta=1$ corresponds to null virial excess, $2\Delta E_{\rm rdm}=2(\tilde{E}_{\rm rdm}-E_{\rm rdm})$, which does not necessarily imply a relaxed configuration[5], $\zeta>1$ to positive virial excess, and $\zeta<1$ to negative virial excess. The special case, $\zeta_{rr}=\tilde{\zeta}_{rr}$, $\zeta=1$, makes Eqs. (107) reduce to (106).

In summary, Eqs. (106) hold provided instantaneous velocity dispersions, $(\sigma_{11}^2,\sigma_{22}^2,\sigma_{33}^2)$, are replaced by effective velocity dispersions, $(\zeta_{11}\sigma^2,\zeta_{22}\sigma^2,\zeta_{33}\sigma^2)=(\tilde{\sigma}_{11}^2,\tilde{\sigma}_{22}^2,\tilde{\sigma}_{33}^2)$, related to an equilibrium or configuration averaged over a sufficiently long time. In other words, Eqs. (106) represent the virial equilibrium configuration of a mass distribution which coincides with its counterpart related to a selected instantaneous configuration of the system under consideration.

For the sake of simplicity, let us define ideal, self-gravitating fluids, rotating around an axis, a_3, for which Eqs. (107) hold i.e. the assumptions (i), (ii), and (iii) above are valid, as R3 fluids[6].

The combination of Eqs. (107a) and (107b) yields:

$$I_{qq}\overline{\Omega}^2-\frac{\zeta_{qq}}{\zeta_{33}}(E_{\rm pot})_{33}+(E_{\rm pot})_{qq}=0\quad;\qquad q=1,2\quad;\qquad(109)$$

to get further insight, let us express the self potential-energy tensor, $(E_{\rm pot})_{rr}$, and the moment of inertia tensor, I_{rr}, in terms of dimensionless tensors, $\mathcal{P}_{rr}$ and $\mathcal{I}_{rr}$, respectively, as:

$$(E_{\rm pot})_{rr}=-\frac{GM^2}{a}\mathcal{P}_{rr}\quad;\qquad r=1,2,3\quad;\qquad(110\text{a})$$

$$(E_{\rm pot})=-\frac{GM^2}{a}\mathcal{P}\quad;\qquad(110\text{b})$$

$$I_{qq}=Ma^2\mathcal{I}_{qq}\quad;\qquad q=1,2\quad;\qquad(111\text{a})$$

$$I_3=Ma^2\mathcal{I}_3\quad;\qquad(111\text{b})$$

$$a=\left(\frac{S}{2\pi}\right)^{1/3}\quad;\qquad(112)$$

furthermore, let us define the rotation parameter:

$$\upsilon=\frac{a^3\overline{\Omega}^2}{GM}=\frac{\overline{\Omega}^2}{2\pi G\overline{\rho}}\quad;\qquad(113)$$

where $\overline{\rho}=M/S$ is the mean density of the system. In the special case of solid-body rotation ($\overline{\Omega}=\Omega$), Eq. (113) reduces to a notation used for polytropes (e.g. Jeans 1929, Chap. IX, § 232; Chandrasekhar and Leboviz 1962), and in the limit of ellipsoidal homogeneous configurations ($\overline{\rho}=\rho$), Eq. (113) reduces to a notation used for MacLaurin spheroids and Jacobi ellipsoids (e.g. Jeans 1929, Chap. VIII, §§ 189-193; C69, Chap. 5, § 32, Chap. 6, § 39).

Owing to Eqs. (110a), (111a) and (112), Eq. (109) may be formulated in terms of dimensionless parameters, as:

$$(\zeta_{33}\mathcal{P}_{qq}-\zeta_{qq}\mathcal{P}_{33})-\upsilon\zeta_{33}\mathcal{I}_{qq}=0\quad;\qquad q=1,2\quad;\qquad(114)$$

which admits real solutions provided the inequality:

$$\frac{\zeta_{qq}}{\zeta_{33}}\le\frac{\mathcal{P}_{qq}}{\mathcal{P}_{33}}\quad;\qquad q=1,2\quad;\qquad(115)$$

is satisfied, and it is the natural extension to R3 fluids of its counterparts related to axisymmetric, relaxed mass distributions (Wiegandt 1982a,b) and homeoidally striated ellipsoids (C06). Imaginary solutions correspond to imaginary rotation parameters i.e. imaginary rotation, as explained in the next section.

After some algebra the combination of Eqs. (114) yields:

$$\mathcal{I}_{22}(\zeta_{33}\mathcal{P}_{11}-\zeta_{11}\mathcal{P}_{33})=\mathcal{I}_{11}(\zeta_{33}\mathcal{P}_{22}-\zeta_{22}\mathcal{P}_{33})\quad;\qquad(116)$$

or equivalently:

$$\zeta_{33}(\mathcal{I}_{22}\mathcal{P}_{11}-\mathcal{I}_{11}\mathcal{P}_{22})=\mathcal{P}_{33}(\mathcal{I}_{22}\zeta_{11}-\mathcal{I}_{11}\zeta_{22})\quad;\qquad(117)$$

which represent alternative expressions of the constraint related to virial equilibrium.

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[5] For instance, a homogeneous sphere undergoing coherent oscillations exhibits $\zeta>1$ at expansion turnover and $\zeta<1$ at contraction turnover. Then there necessarily exists a configuration where $\zeta=1$ which, on the other hand, is unrelaxed.

[6] As explained earlier, the term "R3" stands for "ideal, self-gravitating fluids, rotating around a principal axis of inertia, let it be x_3". In the general case of R fluids, (systematic) rotation may occur also around the remaining principal axes of inertia, x_2 and/or x_1.

3.4. Axisymmetric and triaxial configurations

An explicit expression for the rotation parameter υ can be derived from Eqs. (114), as:

$$\upsilon = \frac{\zeta_{33}\mathcal{P}_{qq} - \zeta_{qq}\mathcal{P}_{33}}{\zeta_{33}\mathcal{I}_{qq}} \quad ; \qquad q = 1, 2 \quad ; \qquad (118)$$

which, in turn, allows an explicit expression of anisotropy parameter ratios ζ_{qq}/ζ_{pp} as:

$$\frac{\zeta_{qq}}{\zeta_{33}} = \frac{\mathcal{P}_{qq}}{\mathcal{P}_{33}}\left[1 - \upsilon\frac{\mathcal{I}_{qq}}{\mathcal{P}_{qq}}\right] \quad ; \quad q = 1, 2 \quad ; \quad (119)$$

$$\frac{\zeta_{11}}{\zeta_{22}} = \frac{\mathcal{P}_{11} - \upsilon\mathcal{I}_{11}}{\mathcal{P}_{22} - \upsilon\mathcal{I}_{22}} \quad ; \qquad (120)$$

and the combination of Eqs. (107e) and (119) yields:

$$\frac{\zeta_{33}}{\zeta} = \frac{\mathcal{P}_{33}}{\mathcal{P} - \upsilon\mathcal{I}_3} \quad ; \qquad (121)$$

which provides an alternative expression for Eqs. (118):

$$\upsilon = \frac{\zeta_{33}\mathcal{P} - \zeta\mathcal{P}_{33}}{\zeta_{33}\mathcal{I}_3} \quad ; \qquad (122)$$

that is equivalent to Eq. (114), and then admits real solutions provided inequality (115) is satisfied.

Finally, Eqs. (114) may be combined as:

$$\frac{\mathcal{I}_{11}}{\mathcal{I}_{22}} = \frac{\zeta_{33}\mathcal{P}_{11} - \zeta_{11}\mathcal{P}_{33}}{\zeta_{33}\mathcal{P}_{22} - \zeta_{22}\mathcal{P}_{33}} \quad ; \qquad (123)$$

where it can be seen that Eqs. (120) and (123) are changed into each other, replacing the terms, $\mathcal{P}_{33}\zeta_{qq}/\zeta_{33}$, by the terms, $\upsilon\mathcal{I}_{qq}$, and vice versa. The above results may be reduced to a single statement.

Theorem 1. *Given an R3 fluid, the relation:*

$$\frac{X_{11}}{X_{22}} = \frac{\mathcal{P}_{11} - Y_{11}}{\mathcal{P}_{22} - Y_{22}} \quad ;$$

$$X_{qq} = \frac{\zeta_{qq}}{\zeta_{33}}\mathcal{P}_{33} \;,\; \upsilon\mathcal{I}_{qq} \quad ; \qquad q = 1, 2 \quad ;$$

$$Y_{qq} = \upsilon\mathcal{I}_{qq} \;,\; \frac{\zeta_{qq}}{\zeta_{33}}\mathcal{P}_{33} \quad ; \qquad q = 1, 2 \quad ;$$

is symmetric with respect to X_{qq} and Y_{qq}, the former tensor being related to anisotropic random velocity distribution, and the latter to systematic rotation around the axis x_3.

In the special case of axisymmetric configurations, the dimensionless factors appearing in the expression of the self potential-energy tensor, Eqs. (110a), and the moment of inertia, Eqs. (111a), do coincide with regard to equatorial axes, $\mathcal{P}_{11} = \mathcal{P}_{22}$ and $\mathcal{I}_{11} = \mathcal{I}_{22}$, respectively, which necessarily imply $\zeta_{11} = \zeta_{22}$, owing to Eq. (123).

In the general case of triaxial configurations, the contrary holds, $\mathcal{P}_{11} \neq \mathcal{P}_{22}$ and $\mathcal{I}_{11} \neq \mathcal{I}_{22}$, and the equality, $\zeta_{11} = \zeta_{22}$, via Eq. (120), implies the validity of the relation:

$$\upsilon = \frac{\mathcal{P}_{11} - \mathcal{P}_{22}}{\mathcal{I}_{11} - \mathcal{I}_{22}} \quad ; \qquad (124)$$

in the opposite case, the random velocity distribution along the equatorial plane[7] is anisotropic i.e. $\zeta_{11} \neq \zeta_{22}$. The related degeneracy can be removed using an additional condition, as it will be shown in the next section.

The above results may be reduced to a single statement.

Theorem 2. *Isotropic random velocity distribution along the equatorial plane, $\zeta_{11} = \zeta_{22}$, makes a necessary condition for R3 fluids to be symmetric with respect to the rotation axis, x_3.*

4. IMAGINARY ROTATION

A unified theory of systematic and random motions is possible, if one introduces imaginary rotation. It was shown above that Eq. (114), or equivalently one among (118), (122), admits real solutions provided inequality (115) is satisfied. If otherwise, the rotation parameter, υ, has necessarily to be negative, which implies, via Eq. (113), an *imaginary* figure rotation, $i\Omega$, where i is the imaginary unit. Accordingly, the related centrifugal potential takes the general expression:

$$\mathcal{T}(x_1, x_2, x_3, t) = \frac{1}{2}\,Sgn\left(\frac{\mathcal{P}_{qq}}{\mathcal{P}_{33}} - \frac{\zeta_{qq}}{\zeta_{33}}\right)$$
$$\times[\overline{\Omega}(x_1, x_2, x_3, t)]^2 w^2 \;;\quad w^2 = x_1^2 + x_2^2 \;;\; (125)$$

where Sgn is the sign function, $Sgn(\mp|x|) = \mp 1$, $Sgn(0) = 0$. The centrifugal force, $\partial\mathcal{T}/\partial w$, is positive or negative depending on whether real or imaginary rotation occurs, respectively. Then the net effect of real rotation is flattening, while that of imaginary rotation is *elongation*, with respect to the rotation axis (Caimmi 1996b, C06).

To get further insight, let us particularize Eq. (114) to the special case of null rotation ($\upsilon = 0$). The result is:

$$\frac{\zeta_{qq}}{\zeta_{33}} = \frac{\mathcal{P}_{qq}}{\mathcal{P}_{33}} \quad ; \qquad \upsilon = 0 \quad ; \qquad q = 1, 2 \quad ; \qquad (126)$$

where the right-hand side, via Eqs. (31) and (110a), depends on the mass distribution only. Accordingly, the net effect of positive ($\zeta_{qq}/\zeta_{33} > 0$) or negative ($\zeta_{qq}/\zeta_{33} < 0$) random motion excess along the equatorial plane is flattening ($\mathcal{P}_{qq} > \mathcal{P}_{33}$) or elongation

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[7] Throughout this paper, "along the equatorial plane" has to be understood as "along any direction parallel to the equatorial plane".

($\mathcal{P}_{qq} < \mathcal{P}_{33}$), respectively. In what follows, it shall be understood that random motion excess is related to the equatorial plane.

4.1. Random motion excess and rotation

In the limit of isotropic random velocity distribution, $\zeta_{11} = \zeta_{22} = \zeta_{33} = \zeta/3$, Eqs. (118) and (122) reduce to:

$$\upsilon_{\rm iso} = \frac{\mathcal{P}_{qq} - \mathcal{P}_{33}}{\mathcal{I}_{qq}} \quad ; \qquad q = 1, 2 \quad ; \tag{127}$$

$$\upsilon_{\rm iso} = \frac{\mathcal{P} - 3\mathcal{P}_{33}}{\mathcal{I}_3} \quad ; \tag{128}$$

where the index, iso, emphasises the isotropic random velocity distribution.

Accordingly, Eqs. (118) and (122) may be expressed as:

$$\upsilon = \upsilon_{\rm iso} - \upsilon_{\rm ani} \quad ; \tag{129}$$

$$\upsilon_{\rm ani} = \left(\frac{\zeta_{qq}}{\zeta_{33}} - 1\right)\frac{\mathcal{P}_{33}}{\mathcal{I}_{qq}} \quad ; \qquad q = 1, 2 \quad ; \tag{130}$$

$$\upsilon_{\rm ani} = \left(\frac{\zeta}{\zeta_{33}} - 3\right)\frac{\mathcal{P}_{33}}{\mathcal{I}_3} \quad ; \tag{131}$$

where $\upsilon_{\rm ani} \geq 0$ for oblate-like configurations, $\zeta_{qq}/\zeta_{33} \geq 1$; $\upsilon_{\rm ani} \leq 0$ for prolate-like configurations, $\zeta_{qq}/\zeta_{33} \leq 1$; and the index, ani, means contribution from random motion excess which, in general, makes for an anisotropic random velocity distribution. Accordingly, positive or negative random motion excess is related to real or imaginary rotation respectively.

Let us rewrite Eq. (129) as:

$$\upsilon_{\rm iso} = \upsilon + \upsilon_{\rm ani} \quad ; \tag{132}$$

which, owing to Eq. (113), is equivalent to:

$$\overline{\Omega}^2_{\rm iso} = \overline{\Omega}^2 + Sgn\left(\frac{\zeta}{\zeta_{33}} - 3\right)\overline{\Omega}^2_{\rm ani} \quad ; \tag{133}$$

where positive and negative Sgn values correspond to real and imaginary rotation, respectively. Then the effect of random motion excess on the shape of the system is virtually indistinguishable from the effect of additional figure rotation. The above results may be reduced to a single statement.

Theorem 3. *Given an R3 fluid, the effect of (positive or negative) random motion excess is equivalent to an additional (real or imaginary) figure rotation, $Sgn(\zeta/\zeta_{33} - 3)\overline{\Omega}^2_{\rm ani}$, with regard to an adjoint configuration where the random velocity distribution is isotropic.*

Accordingly, an R3 fluid with assigned systematic rotation and random velocity distribution, as far as shape is concerned, is virtually indistinguishable from an adjoint configuration of equal density profile, isotropic random velocity distribution, and figure rotation deduced from Eq. (133).

4.2. Axisymmetric and triaxial configurations

The combination of alternative expressions for the rotation parameter, $\upsilon_{\rm iso}$, defined by Eqs. (127), yields:

$$\mathcal{I}_{11}\mathcal{P}_{22} - \mathcal{I}_{22}\mathcal{P}_{11} = \mathcal{P}_{33}(\mathcal{I}_{11} - \mathcal{I}_{22}) \quad ; \tag{134}$$

which, for axisymmetric configurations, i.e. for $\mathcal{I}_{11} = \mathcal{I}_{22}$ and $\mathcal{P}_{11} = \mathcal{P}_{22}$, reduces to an indeterminate form, $0 = 0$.

The combination of alternative expressions for the rotation parameter, $\upsilon_{\rm ani}$, defined by Eq. (129), yields:

$$\mathcal{I}_{11}\zeta_{22} - \mathcal{I}_{22}\zeta_{11} = \zeta_{33}(\mathcal{I}_{11} - \mathcal{I}_{22}) \quad ; \tag{135}$$

which, for isotropic random velocity distributions, reduces to an indeterminate form, $0 = 0$. In addition, axisymmetric configurations ($\mathcal{I}_{11} = \mathcal{I}_{22}$) necessarily imply isotropic random velocity distributions along the equatorial plane, $\zeta_{11} = \zeta_{22}$.

The combination of Eqs. (107e) and (135) yields:

$$\zeta_{qq} = \frac{\zeta\mathcal{I}_{qq} - \zeta_{33}(2\mathcal{I}_{qq} - \mathcal{I}_{pp})}{\mathcal{I}_3} \quad ; \qquad q = 1, 2 \quad ; \qquad p = 2, 1 \quad ; \tag{136}$$

which, for axisymmetric configurations ($\mathcal{I}_{11} = \mathcal{I}_{22} = \mathcal{I}_3/2$) reduces to :

$$\zeta_{qq} = \frac{\mathcal{I}_{qq}}{\mathcal{I}_3}(\zeta - \zeta_{33}) = \frac{\zeta - \zeta_{33}}{2} \quad ; \qquad q = 1, 2 \quad ; \tag{137}$$

and the special case, $\zeta_{33} = \zeta/3$, reads $\zeta_{11} = \zeta_{22} = \zeta/3$.

The limiting configuration, $\zeta_{qq} = 0$, via Eqs. (134), necessarily implies $\mathcal{I}_{pp} \leq \mathcal{I}_{qq}$, owing to $\zeta_{pp} \geq 0$, $q = 1, 2$, $p = 2, 1$, and Eq. (136) reduces to:

$$\zeta\mathcal{I}_{qq} - \zeta_{33}(2\mathcal{I}_{qq} - \mathcal{I}_{pp}) = 0 \quad ; \qquad q = 1, 2 \quad ; \qquad p = 2, 1 \quad ; \tag{138}$$

which, owing to Eq. (107e), is equivalent to:

$$\frac{\mathcal{I}_{pp}}{\mathcal{I}_{qq}} = \frac{\zeta_{33} - \zeta_{pp}}{\zeta_{33}} = \frac{2\zeta_{33} - \zeta}{\zeta_{33}} \quad ; \tag{139a}$$

$$\zeta_{qq} = 0 \quad ; \qquad q = 1, 2 \quad ; \qquad p = 2, 1 \quad ; \tag{139b}$$

where $\mathcal{I}_{qq}/\mathcal{I}_{pp} \geq 1$ implies $\zeta_{33} \geq \zeta_{pp}$ and $\zeta_{33} \geq \zeta/2$. The above results may be reduced to the following statements.

Theorem 4. *Given an R3 fluid, the anisotropy parameters along the equatorial plane, ζ_{qq}, $q = 1, 2$, depend on the diagonal components of the dimensionless moment of inertia tensor, $\mathcal{I}_{qq}$, $q = 1, 2$, and the related expressions coincide, $\zeta_{11} = \zeta_{22}$, in the limit of axisymmetric configurations, $\mathcal{I}_{11} = \mathcal{I}_{22}$.*

Theorem 5. *Given an R3 fluid, a necessary and sufficient condition for isotropic random* *velocity distribution is that the anisotropy parameter along the rotation axis attains the value, $\zeta_{33} = 1/3$.*

Theorem 6. *Given a sequence of R3 fluids, the ending point occurs when the third diagonal component of the dimensionless self potential-energy tensor is zero, $\mathcal{P}_{33} = 0$, and/or the generalized anisotropy parameter related to the major equatorial axis is zero, $\zeta_{11} = 0$, which is equivalent to $\mathcal{I}_{22}/\mathcal{I}_{11} = (2 - \zeta/\zeta_{33})^{1/2}$. The related value of the rotation parameter is $\upsilon = \mathcal{P}_{qq}/\mathcal{I}_{qq}$, $q = 1, 2$, independent of anisotropy parameters. The special case of dynamical (or hydrostatic) equilibrium, $\zeta = 1$, implies centrifugal support along the major equatorial axis, provided $\zeta_{11} = 0$.*

Accordingly, for the R3 fluids, the anisotropy parameters along the equatorial plane, ζ_{11} and ζ_{22}, cannot be arbitrarily chosen, but depend on the dimensionless moment of inertia tensor diagonal components, $\mathcal{I}_{11}$ and $\mathcal{I}_{22}$, according to Eqs. (136). On the other hand, the knowledge of the dimensionless moment of inertia tensor components, $\mathcal{I}_{11}$ and $\mathcal{I}_{22}$, the dimensionless self potential-energy tensor components, $\mathcal{P}_{11}$, $\mathcal{P}_{22}$ and $\mathcal{P}_{33}$, together with the rotation parameter, υ, allows the determination of the rotation parameter, $\upsilon_{\rm ani}$, via Eqs. (127), (128), (129), and then the ratios, ζ_{qq}/ζ_{33}, ζ_{33}/ζ, via Eqs. (130), (131), respectively, or the anisotropy parameter along the rotation axis, ζ_{33}, provided the virial index, ζ, defined by Eq. (107e), is assigned.

In conclusion, for the R3 fluids defined by specified dimensionless moment of inertia tensor components, $\mathcal{I}_{11}$ and $\mathcal{I}_{22}$, dimensionless self potential-energy tensor components, $\mathcal{P}_{11}$, $\mathcal{P}_{22}$ and $\mathcal{P}_{33}$, rotation parameter, υ, and virial index, ζ, the anisotropy parameters, ζ_{11}, ζ_{22}, ζ_{33}, cannot be arbitrarily chosen, but must be determined as shown above.

4.3. Sequences of virial equilibrium configurations

For the R3 fluids, it has been shown above that corresponding configurations are characterized by (i) centrifugal potential, $\mathcal{T}_{\rm iso}(x_1, x_2, x_3) = \mathcal{T}(x_1, x_2, x_3) + \mathcal{T}_{\rm ani}(x_1, x_2, x_3)$, or $\overline{\Omega}^2_{\rm iso}(x_1, x_2, x_3) = \overline{\Omega}^2(x_1, x_2, x_3) + Sgn(\zeta/\zeta_{33} - 3)\overline{\Omega}^2_{\rm ani}(x_1, x_2, x_3)$, Eq. (133); and (ii) isotropic random velocity distribution. Owing to Theorem 3, a sequence of R3 fluids coincides with the sequence of adjoint configurations. Given an R3 fluid with fixed components of the dimensionless self potential-energy tensor, $\mathcal{P}_{11}$, $\mathcal{P}_{22}$, $\mathcal{P}_{33}$, the dimensionless moment of inertia tensor, $\mathcal{I}_{11}$, $\mathcal{I}_{22}$, the rotation parameter, υ, and virial index, ζ, the anisotropy parameters, ζ_{11}, ζ_{22}, ζ_{33}, are determined via Eqs. (127)-(131) and (136). Negative values of the rotation parameter, $\upsilon_{\rm iso}$, extend the sequence of axisymmetric configurations to imaginary rotation, i.e. prolate configurations where the major axis coincides with the rotation axis.

As in sequences of homeoidally striated Jacobi ellipsoids (C06), the meridional axis ratio lies in the range, $0 \le \epsilon_{31} \le 1$, for real rotation, and $\epsilon_{31} > 1$, for imaginary rotation, also in sequences of R3 fluids, where a bifurcation point from axisymmetric to triaxial configurations is expected to occur when the meridional axis ratio attains a threshold value. The location of an R3 fluid in a selected sequence allows the knowledge of a number of physical parameters, such as shape, angular momentum, moment of inertia tensor, self potential-energy tensor, and kinetic-energy tensor.

Once more owing to Theorem 3, the bifurcation point of a sequence of R3 fluids coincides with its counterpart along the sequence of adjoint configurations. Aiming at finding a necessary condition for the occurrence of a bifurcation point, let us equate the alternative expressions of Eqs. (118). The result is:

$$\frac{\mathcal{P}_{11}}{\mathcal{I}_{11}} - \frac{\zeta_{11}}{\zeta_{33}}\frac{\mathcal{P}_{33}}{\mathcal{I}_{11}} = \frac{\mathcal{P}_{22}}{\mathcal{I}_{22}} - \frac{\zeta_{22}}{\zeta_{33}}\frac{\mathcal{P}_{33}}{\mathcal{I}_{22}} \quad ; \tag{140}$$

where the anisotropy parameter ratios ζ_{qq}/ζ_{33}, $q = 1, 2$, may be deduced from Eqs. (136). After some algebra, Eq. (140) becomes:

$$\frac{\mathcal{I}_{11}\mathcal{P}_{22} - \mathcal{I}_{22}\mathcal{P}_{11}}{\mathcal{I}_{11} - \mathcal{I}_{22}} = \mathcal{P}_{33} \quad ; \tag{141}$$

and the occurrence of a bifurcation point has necessarily to satisfy the relation:

$$\lim_{\epsilon_{21}\to 1} \frac{\mathcal{I}_{11}\mathcal{P}_{22} - \mathcal{I}_{22}\mathcal{P}_{11}}{\mathcal{I}_{11} - \mathcal{I}_{22}} = \mathcal{P}_{33} \quad ; \tag{142}$$

where $\epsilon_{21} = a_2/a_1$ is the ratio of two generic equatorial (perpendicular) radii, and $\epsilon_{21} \to 1$ implies $\mathcal{I}_{22} \to \mathcal{I}_{11}$, $\mathcal{P}_{22} \to \mathcal{P}_{11}$. Then Eq. (142) is a necessary condition for the existence of a bifurcation point, as it selects the sole axisymmetric configuration which satisfies Eq. (141), regardless from the values of the anisotropy parameters, ζ_{rr}, $r = 1, 2, 3$. The above results may be reduced to a single statement.

Theorem 7. *Given a sequence of R3 fluids, a necessary condition for the existence of a bifurcation point from axisymmetric to triaxial configurations, is independent of the anisotropy parameters, ζ_{rr}, $r = 1, 2, 3$, and coincides with its counterpart related to the sequence of adjoint configurations with isotropic random velocity distribution.*

Sequences of R3 fluids with assigned density profiles, can be deduced from the knowledge of the rotation parameters, $\upsilon_{\rm iso}(\epsilon_{31})$ and $\upsilon_{\rm ani}(\epsilon_{31}, \tilde{\zeta}_{33})$, as functions of the meridional axis ratio, $\epsilon_{31} = a_3/a_1$, and the effective anisotropy parameter, $\tilde{\zeta}_{33}$, as represented in Figs. 2 and 3, respectively[8].

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[8] Strictly speaking, the curves plotted in Figs. 2, 3, and 4 are related to the special case of homeoidally striated Jacobi ellipsoids (C06), but can be taken as illustrative for R3 fluids.

A hypothetical sequence of axisymmetric R3 fluids, extended to the case of imaginary rotation, is shown in Fig. 2. Hypothetical dependences of the rotation parameter, $\upsilon_{\rm ani}$, on the meridional axis ratio, ϵ_{31}, and the effective anisotropy parameter, $\tilde{\zeta}_{33} = \zeta_{33}/\zeta$ (labelled on each curve), are shown in Fig. 3.

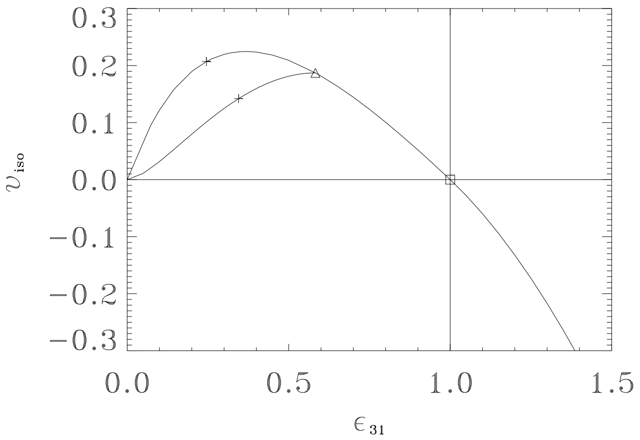

Fig. 2. *A hypothetical sequence of axisymmetric R3 fluids, from the starting point (square) to the bifurcation point (triangle), and related triaxial R3 fluids, from the starting point (triangle) to the bifurcation point (Greek cross), extended to the case of imaginary rotation (negative values of the rotation parameter, υ). In any case, the random velocity distribution is isotropic. Both sequences are continued in the region of instability. The horizontal line, $\upsilon_{\rm iso} = 0$, is the locus of non rotating and/or zero volume configurations. The vertical line, $\epsilon_{31} = 1$, is the locus of round ($a_1 = a_2 = a_3$) configurations. The above mentioned lines divide the non negative semiplane, $\epsilon_{31} \geq 0$, into four regions (from top left in clockwise sense): A - oblate-like shapes with real rotation; B - prolate-like shapes with real rotation; C - prolate-like shapes with imaginary rotation; D - oblate-like shapes with imaginary rotation. Regions B and D are forbidden to sequences of R3 fluids.*

With regard to a fixed effective anisotropy parameter, the related sequence starts from a nonrotating configuration, $\upsilon = 0$, i.e. $\upsilon_{\rm ani} = \upsilon_{\rm iso}$, and ends at a configuration where $\epsilon_{31} = 0$ and/or $\tilde{\zeta}_{11} = 0$. A hypothetical locus of the ending points related to the latter alternative, is represented as a long-dashed curve. The locus of nonrotating configurations (short-dashed lines) coincides with the curves represented in Fig. 2. No sequence can be continued on the right, as imaginary rotation i.e. larger $\tilde{\zeta}_{33}$ would be needed and a different sequence should be considered. The effect of positive ($\tilde{\zeta}_{33} < 1/3$) or negative ($\tilde{\zeta}_{33} > 1/3$) random motion excess is equivalent to an additional real or imaginary rotation, respectively. The horizontal non negative semiaxis, $\epsilon_{31} \geq 0$, $\upsilon_{\rm ani} = 0$, is the locus of configurations with isotropic random velocity distribution, $\tilde{\zeta}_{33} = 1/3$. The vertical straight line, $\epsilon_{31} = 1$, is the locus of round ($a_1 = a_2 = a_3$) configurations.

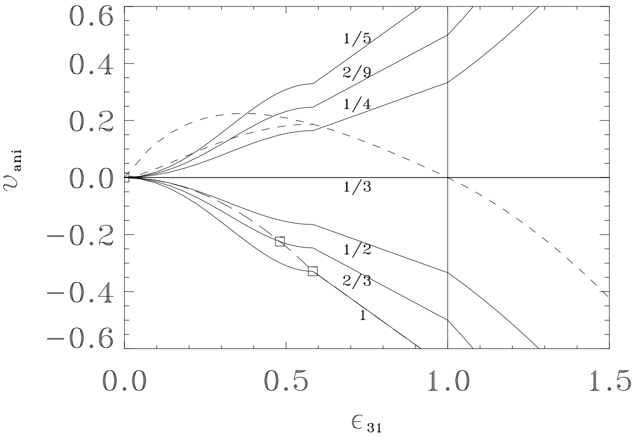

Fig. 3. *A hypothethical dependence of the rotation parameter, $\upsilon_{\rm ani}$, related to random motion excess, on the meridional axis ratio, ϵ_{31}, with regard to R3 fluids. Each curve is labelled by the corresponding value of the effective anisotropy parameter, $\tilde{\zeta}_{33} = \zeta_{33}/\zeta$. The horizontal non negative semiaxis, $\epsilon_{31} \geq 0$, $\upsilon_{\rm ani} = 0$, is the locus of configurations with isotropic random velocity distribution, $\tilde{\zeta}_{33} = 1/3$. The vertical straight line, $\epsilon_{31} = 1$, is the locus of round ($a_1 = a_2 = a_3$) configurations. The generic sequence starts from a non rotating configuration (short-dashed lines) and ends at a configuration where either $\epsilon_{31} = 0$ and/or $\tilde{\zeta}_{11} = 0$ (long-dashed line). The regions above the upper short-dashed curve and below the long-dashed curve, respectively, are forbidden to R3 fluids. The non negative vertical semiaxis, $\upsilon_{\rm ani} \geq 0$, $\epsilon_{31} = 0$, corresponds to flat ($\tilde{\zeta}_{33} = 0$) configurations with no figure rotation. The curves are symmetric with respect to the horizontal axis, until the limiting curve, $\tilde{\zeta}_{33} = \tilde{\zeta} = 1$, is reached. The limiting configuration where $\tilde{\zeta}_{11} = 0$, is marked by a square: no configuration in virial equilibrium is allowed on the left, as it would imply $\tilde{\zeta}_{11} < 0$. No configuration is allowed on the right of the starting point, as it would imply $\upsilon < 0$.*

With regard to real rotation, the ending configuration ($\epsilon_{31} = 0$ and/or $\tilde{\zeta}_{11} = 0$) is marked by a square. Configurations on the left are forbidden, as they would imply negative random energy tensor component, $(E_{\rm rdm})_{11} < 0$, to satisfy the virial equation (107a), which requires imaginary rotation around major equatorial axis, i.e. systematic motions other than rotation around the minor axis.

With regard to imaginary rotation, no ending point occurs and the system is allowed to attain negative infinite rotation parameter, $\upsilon_{\rm iso} \to -\infty$, and infinite meridional axis ratio, $\epsilon_{31} \to +\infty$. The related configuration is either a spindle ($a_1 = a_2 = 0$) or an infinitely high cylinder ($a_1 = a_2 < a_3 \to +\infty$).

Further inspection of Fig. 3. shows additional features, namely: (i) zero left first derivatives on each sequence at bifurcation point (not marked therein for sake of clarity), and (ii) occurrence of symmetric sequences with respect to the horizontal axis (including also forbidden configurations). For additional considerations on (i), see C06 (nothing changes with respect to the special case investigated therein). The above results may be reduced to a single statement.

> **Theorem 8.** *Given a sequence of R3 fluids, the contribution of random motion excess, $\upsilon_{\rm ani}$, to the rotation parameter, $\upsilon_{\rm iso}$, has a zero left first derivative at the bifurcation point, $[d\upsilon_{\rm ani}/d\epsilon_{31}]_{(\epsilon_{31})^{-}_{\rm bif}} = 0$.*

The occurrence of symmetric sequences with respect to the horizontal axis, is deduced from Eq. (131), using the condition:

$$(\tilde{\zeta}_{33}^{+})^{-1} - 3 = -(\tilde{\zeta}_{33}^{-})^{-1} + 3 \quad ; \qquad (143)$$

where $\tilde{\zeta}_{33}^{\mp} = \zeta_{33}^{\mp}/\zeta$ is related to curves characterized by negative ($\tilde{\zeta}_{33} = \tilde{\zeta}_{33}^{-} \ge 1/3$) and positive ($\tilde{\zeta}_{33} = \tilde{\zeta}_{33}^{+} \le 1/3$) values, respectively, of the rotation parameter, $\upsilon_{\rm ani}$, see Fig. 3. Couples of symmetric sequences (including forbidden configurations) start from $(\tilde{\zeta}_{33}^{-}, \tilde{\zeta}_{33}^{+}) = (1/3, 1/3)$, where each curve coincides with the other, and end at (1,1/5). Sequences in the range, $0 \le \tilde{\zeta}_{33}^{+} < 1/5$, have no symmetric counterpart.

Let a point, $\mathsf{P}[\epsilon_{31}, \upsilon_{\rm iso}]$, be fixed on a sequence of axisymmetric R3 fluids, and the dimensionless moment of inertia tensor components, $\mathcal{I}_{11}$ and $\mathcal{I}_{22}$, be determined by use of Eqs. (127) and (128). Let a point, $\mathsf{P}'(\epsilon_{31}, \upsilon_{\rm ani})$, be fixed on the plane, ($\mathsf{O}\ \epsilon_{31}\ \upsilon_{\rm ani}$), and the effective anisotropy parameters, $\tilde{\zeta}_{11}$, $\tilde{\zeta}_{22}$, $\tilde{\zeta}_{33}$, be determined by use of Eqs. (130), (131), (136). Finally, let the rotation parameter, υ, be determined by use of Eq. (129). Following the above procedure, sequences of R3 fluids may be generated. For assigned density profiles and systematic rotation velocity fields, there are three independent parameters, which may be chosen as two axis ratios, ϵ_{21}, ϵ_{31}, and one effective anisotropy parameter, $\tilde{\zeta}_{33}$.

In the ($\mathsf{O}\epsilon_{31}\upsilon$) plane (Fig. 3.), each sequence starts from the intersection between the curves, $\upsilon = [\upsilon(\epsilon_{31})]_{\rm iso}$ (short-dashed), $\upsilon = [\upsilon(\epsilon_{31}; \tilde{\zeta}_{33})]_{\rm ani}$ (full), and ends at the intersection between the curves, $\upsilon = \upsilon(\epsilon_{31}; \tilde{\zeta}_{11} = 0)$ (long-dashed), $\upsilon = [\upsilon(\epsilon_{31}; \tilde{\zeta}_{33})]_{\rm ani}$ (full).

Hypothetical dependences of the rotation parameter, υ, on the meridional axis ratio, ϵ_{31}, and the effective anisotropy parameter, $\tilde{\zeta}_{33} = \zeta_{33}/\zeta$ (same cases as in Fig. 3.), are shown in Fig. 4.

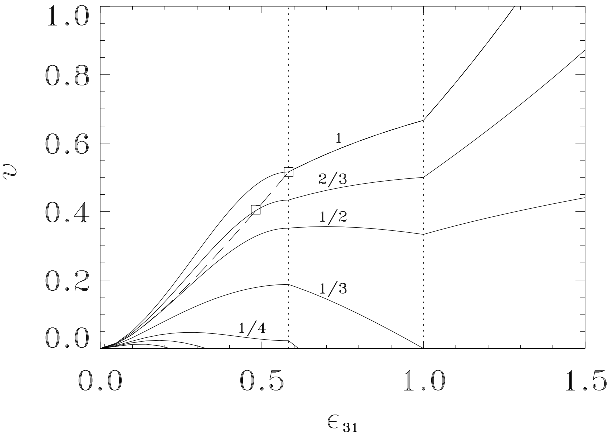

Fig. 4. *A hypothethical dependence of the rotation parameter, υ, related to systematic rotation, on the meridional axis ratio, ϵ_{31}, with regard to R3 fluids. Each curve is labelled by the value of the effective anisotropy parameter, $\tilde{\zeta}_{33} = \zeta_{33}/\zeta$, except the lowermost two, where $\tilde{\zeta}_{33} = 2/9$ and $1/5$, respectively. The dotted vertical straight lines denote a hypothetical bifurcation point (left) and the round ($a_1 = a_2 = a_3$) configuration. The generic sequence starts from a non rotating configuration on the horizontal axis and ends at a configuration where $\epsilon_{31} = 0$ and/or $\tilde{\zeta}_{11} = 0$ (long-dashed line), denoted by a square. The continuation on the left of the ending point, where $\tilde{\zeta}_{11} < 0$, $1/2 < \tilde{\zeta}_{33} \le 1$, is also shown. The initial configuration, related to $\tilde{\zeta}_{33} = 1$, corresponds to $0 = a_1 = a_2 < a_3$ or $a_1 = a_2 < a_3 \to +\infty$, which is equivalent to $\epsilon_{31} \to +\infty$.*

A hypothetical locus of the ending points related to $\tilde{\zeta}_{11} = 0$ is represented as a long-dashed curve, corresponding to $1/2 < \tilde{\zeta}_{33} \le 1$. The continuation of a generic sequence on the left of the long-dashed curve would imply $\tilde{\zeta}_{11} < 0$ or $\epsilon_{31} < 0$. The ending point of sequences, related to $0 \le \tilde{\zeta}_{33} < 1/2$, coincides with the origin. The initial configuration, related to $\tilde{\zeta}_{33} = 1$, corresponds to $0 = a_1 = a_2 < a_3$ or $a_1 = a_2 < a_3 \to +\infty$, which is equivalent to $\epsilon_{31} \to +\infty$.

4.4. A special case: homeoidally striated Jacobi ellipsoids

Homeoidally striated Jacobi ellipsoids are a special case of R3 fluids, for which the results are already known (CM05, C06). The particularization of the current theory to homeoidally striated Jacobi ellipsoids makes useful check. In the case under discussion, Eqs. (110a) and (112) reduce to (e.g. CM05, C06):

$$(E_{\rm pot})_{rr} = -\frac{GM^2}{a_1}\nu_{\rm sel}\epsilon_{r2}\epsilon_{r3}A_r \quad ; \qquad r = 1, 2, 3 \; ; \; (144)$$

$$\mathcal{P}_{rr}=\left(\frac{2}{3}\right)^{1/3}\nu_{\rm sel}(\epsilon_{21}\epsilon_{31})^{1/3}\epsilon_{r2}\epsilon_{r3}A_r\quad; \tag{145}$$

$$a=\left(\frac{2}{3}\right)^{1/3}a_1(\epsilon_{21}\epsilon_{31})^{1/3}\quad; \tag{146}$$

where $\nu_{\rm sel}$ is a profile factor, i.e. depends only on the radial density profile, A_r, $r=1,2,3$, are shape factors, i.e. depend on the axis ratios only, and a_r, $r=1,2,3$, are the semiaxes of the ellipsoidal boundary. The dimensionless density profile may be represented as:

$$\rho=\rho_0 f(\xi)\quad;\quad f(1)=1\quad;\quad \rho_0=\rho(1)\quad; \tag{147a}$$

$$\xi=\frac{r}{r_0}\quad;\quad 0\le\xi\le\Xi\quad;\quad \Xi=\frac{R}{r_0}\quad; \tag{147b}$$

where ρ_0, r_0, are a scaling density and a scaling radius, respectively, with respect to a reference isopycnic surface, while $\Xi=\Xi(R,\theta,\phi)$, and R are related to the truncation isopycnic surface.

The mass, M, and the moment of inertia tensor, I_{pq}, are (e.g. CM05):

$$M=\nu_{\rm mas}M_0\quad; \tag{148}$$

$$I_{pq}=\delta_{pq}\nu_{\rm inr}Ma_p^2\quad; \tag{149}$$

where M_0 is the mass of a homogeneous ellipsoid with same density and boundary as the reference isopycnic surface, whereas $\nu_{\rm mas}$ and $\nu_{\rm inr}$ are profile factors.

The combination of Eqs. (102), (111a), (112), (147b), and (149) yields:

$$\mathcal{I}_{qq}=\frac{I_{qq}}{Ma^2}=\nu_{\rm inr}\left[\frac{2}{3}\epsilon_{21}\epsilon_{31}\right]^{-2/3}\epsilon_{q1}^2\quad; \tag{150}$$

$$\mathcal{I}_3=\mathcal{I}_{11}+\mathcal{I}_{22}=\nu_{\rm inr}\left[\frac{2}{3}\epsilon_{21}\epsilon_{31}\right]^{-2/3}(1+\epsilon_{21}^2); \tag{151}$$

which allows the particularization of the general results pertaining to R3 fluids, to homeoidally striated Jacobi ellipsoids.

Using Eqs. (118), (145), (150), and (151), and performing some algebra, the rotation parameter υ takes the expression:

$$\upsilon=\frac{2}{3}\frac{\nu_{\rm sel}}{\nu_{\rm inr}}\frac{\zeta_{33}A_q-\zeta_{qq}\epsilon_{3q}^2A_3}{\zeta_{33}}\quad;\qquad q=1,2\quad. \tag{152}$$

Let us define a normalized rotation parameter:

$$\upsilon_{\rm N}=\frac{3}{2}\frac{\nu_{\rm inr}}{\nu_{\rm sel}}\upsilon\quad; \tag{153}$$

and re-write Eq. (152) as:

$$\upsilon_{\rm N}=\frac{\zeta_{33}A_q-\zeta_{qq}\epsilon_{3q}^2A_3}{\zeta_{33}}\quad;\qquad q=1,2\quad; \tag{154}$$

which, in spite of a different definition of the rotation parameter, υ, coincides with a previously known result (C06), i.e. $\upsilon_{\rm N}=(\upsilon_{\rm N})_{\rm C06}$.

Using Eqs. (119), (146), (150), the anisotropy parameter ratio, ζ_{qq}/ζ_{33}, assumes the form:

$$\frac{\zeta_{qq}}{\zeta_{33}}=\epsilon_{q3}^2\frac{A_q}{A_3}\left(1-\frac{\upsilon_{\rm N}}{A_q}\right)\quad;\qquad q=1,2\quad; \tag{155}$$

which, keeping in mind a different definition of the rotation parameter, can be shown to coincide with a previously known result (C06). In addition, Eqs. (155) reveal that:

$$\frac{\zeta_{22}}{\zeta_{11}}=\epsilon_{21}^2\frac{A_2-\upsilon_{\rm N}}{A_1-\upsilon_{\rm N}}\quad; \tag{156}$$

which also has necessarily to coincide with a previously known result (C06).

Using Eqs. (122), (146), (151), the alternative expression of the rotation parameter, υ, takes the form:

$$\upsilon_{\rm N}=\frac{\zeta_{33}(A_1+\epsilon_{21}^2A_2+\epsilon_{31}^2A_3)-\zeta\epsilon_{31}^2A_3}{(1+\epsilon_{21}^2)\zeta_{33}}\quad; \tag{157}$$

which, in spite of a different definition of the rotation parameter, υ, coincides with a previously known result (C06) i.e. $\upsilon_{\rm N}=(\upsilon_{\rm N})_{\rm C06}$.

Using Eqs. (123), (145), (151), the dimensionless moment of inertia tensor component ratio, $\mathcal{I}_{22}/\mathcal{I}_{11}$, becomes:

$$\frac{\mathcal{I}_{22}}{\mathcal{I}_{11}}=\epsilon_{21}^2=\frac{\zeta_{33}\epsilon_{21}^2A_2-\zeta_{22}\epsilon_{31}^2A_3}{\zeta_{33}A_1-\zeta_{11}\epsilon_{31}^2A_3}\quad; \tag{158}$$

which, can be shown to coincide with a previously known result (C06; the counterpart of $\mathcal{I}_{22}/\mathcal{I}_{11}$ is $\mathcal{R}_{22}/\mathcal{R}_{11}$ therein).

Using Eqs. (124), (145), (150), (153), the rotation parameter, υ, related to isotropic random velocity distribution along the equatorial plane, $\zeta_{11}=\zeta_{22}$, takes the form:

$$\upsilon_{\rm N}=\frac{A_1-\epsilon_{21}^2A_2}{1-\epsilon_{21}^2}\quad; \tag{159}$$

which, can be shown to coincide with a previously known result (C06).

The above results hold, in particular, for isotropic random velocity distributions ($\zeta_{11}=\zeta_{22}=\zeta_{33}=\zeta/3$), which implies that the expressions of the rotation parameters, $\upsilon_{\rm iso}$ and $\upsilon_{\rm ani}$, via Eq. (132), must necessarily coincide with their previously known counterparts (C06).

Using Eqs. (136), (150), (151), the anisotropy parameters along the equatorial plane, ζ_{qq}, $q=1,2$, become:

$$\zeta_{qq}=\frac{\zeta\epsilon_{q1}^2-\zeta_{33}(2\epsilon_{q1}^2-\epsilon_{2q}^2)}{1+\epsilon_{21}^2}\quad;\qquad q=1,2\quad; \tag{160}$$

which coincide with previously known results (C06), particular for axisymmetric configurations ($\epsilon_{21}=1$).

Using Eqs. (139) and (158), the condition for the occurrence of the ending configuration, related to $\zeta_{11}=0$, takes the form:

$$\epsilon_{21}^2=\frac{\zeta_{33}-\zeta_{22}}{\zeta_{33}}=\frac{2\zeta_{33}-\zeta}{\zeta_{33}}\quad ;\qquad \zeta_{11}=0\quad ;\quad (161)$$

which coincides with a previously known result (C06).

Using Eqs. (141), (145), (150), a general relation between dimensionless self potential-energy tensor components and dimensionless moment of inertia tensor components, takes the form:

$$\frac{\epsilon_{21}^2(A_2-A_1)}{(1-\epsilon_{21}^2)}=\epsilon_{31}^2A_3\quad ;\qquad (162)$$

which coincides with a previously known result (Caimmi 1996a, C06). Accordingly, a necessary condition for the occurrence of a bifurcation point, Eq. (142), reduces to:

$$\lim_{\epsilon_{21}\to 1}\frac{\epsilon_{21}^2(A_2-A_1)}{1-\epsilon_{21}^2}=\epsilon_{31}^2A_3\quad ;\qquad (163)$$

which also coincides with a previously known result (Caimmi 1996a, C06).

5. CONCLUSION

The current paper was aimed at getting more insight on three main points concerning large-scale astrophysical systems, namely: (i) formulation of tensor virial equations from the standpoint of analytical mechanics; (ii) investigation on the role of systematic and random motions with respect to equilibrium configurations; (iii) extent to which systematic and random motions are equivalent in flattening or elongating the shape of a mass distribution.

The tensor virial equations were formulated regardless of the nature of the system and its constituents, by generalizing and extending the procedure used for the scalar virial equations, in presence of discrete subunits (Landau and Lifchitz 1966, Chap. II, § 10). In particular, the self potential-energy tensor was shown to be symmetric with respect to the indices, $(E_{\rm pot})_{pq}=(E_{\rm pot})_{qp}$. Then the results were extended to continuous mass distributions.

The role of systematic and random motions in collisionless, ideal, self-gravitating fluids was analysed in detail including radial and tangential velocity dispersions on the equatorial plane, and the related mean angular velocity, $\overline{\Omega}$, was conceived as a figure rotation.

R3 fluids were defined as ideal, self-gravitating fluids in virial equilibrium, with systematic rotation around a principal axis of inertia, taken to be x_3. The related virial equations have been written in terms of the moment of inertia tensor, I_{pq}, the self potential-energy tensor, $(E_{\rm pot})_{pq}$, and the generalized anisotropy tensor, ζ_{pq} (CM05, C06). Additional effort was devoted to the investigation of the properties of axisymmetric and triaxial configurations.

A unified theory of systematic and random motions was developed for R3 fluids, taking into consideration imaginary rotation (Caimmi 1996b, C06), and a number of theorems previously stated for homeoidally striated Jacobi ellipsoids (C06) were extended to the more general case of R3 fluids. The effect of random motion excess was shown to be equivalent to an additional real or imaginary rotation, inducing flattening (along the equatorial plane) or elongation (along the rotation axis), respectively. Then it was realized that a R3 fluid always admits an adjoint configuration with isotropic random velocity distribution.

In addition, further constraints were established on the amount of random velocity anisotropy along the principal axes, for triaxial configurations. A necessary condition was formulated for the occurrence of bifurcation points from axisymmetric to triaxial configurations in virial equilibrium, which is independent of the anisotropy parameters.

The particularization to the special case of homeoidally striated Jacobi ellipsoid was given, and some previously known results (C06) were reproduced.

APPENDIX

A. Tensor potentials and potential-energy tensors

For reasons of simplicity (integrals are easier than summations to be calculated), let us consider a continuous distribution of matter, where volume elements, ΔS, interact with one another according to their charge density, $\rho_\chi=\Delta\phi_\chi/\Delta S$, with $\Delta\phi_\chi$ the charge within ΔS. The intention is that the results found in this section can be extended to discrete mass distributions, using summations instead of integrals. Let $(\mathsf{O}\ x_1\ x_2\ x_3)$ be a reference frame where the origin coincides with the centre of mass.

The effect of the interaction on an infinitesimal volume element, $d^3S=dx_1\,dx_2\,dx_3$, due to a charge distribution of density, $\rho_\chi(x_1,x_2,x_3)$, is determined by the potential:

$$\mathcal{V}(x_1,x_2,x_3)=G_\chi\int_S\frac{\rho_\chi(x_1^\prime,x_2^\prime,x_3^\prime)\,d^3S^\prime}{\left[\displaystyle\sum_{s=1}^3(x_s-x_s^\prime)^2\right]^{-\chi/2}}\quad ;\qquad (164)$$

where the constants χ and G_χ specify the nature and the intensity of the interaction, respectively, and S is the volume occupied by the system.

The first derivatives of the potential with respect to the coordinates are:

$$\frac{\partial \mathcal{V}}{\partial x_s} = \chi G_\chi \int_S \frac{\rho_\chi(x_1', x_2', x_3')(x_s - x_s')}{\left[\sum_{s=1}^{3}(x_s - x_s')^2\right]^{1-\chi/2}} \, d^3S' \quad ;$$

$$s = 1, 2, 3 \quad ; \qquad (165)$$

it can be seen that the functions of the coordinates, $\mathcal{V}$ and $x_p \, \partial\mathcal{V}/\partial x_q$, $1 \le p \le 3$, $1 \le q \le 3$, are homogeneous functions of degree χ, and the Euler theorem holds (e.g. LL66, Chap. IV, § 10).

With regard to a selected infinitesimal volume element, the potential may be thought of as the tidal potential energy due to the whole charge distribution, related to the point under consideration, with the unit charge placed therein. Associated with the potential, defined by Eqs. (164) and (165), is the self potential energy:

$$E_{\rm pot} = -\frac{1}{2} \int_S \rho_\chi(x_1, x_2, x_3) \mathcal{V}(x_1, x_2, x_3) \, d^3S \quad ; \qquad (166)$$

the self potential energy may be thought of as the tidal potential energy due to the whole charge distribution, related to all the infinitesimal volume elements, provided each pair is counted only once.

The coincidence of Eqs. (166) and (32) may be verified along the following steps: (i) write the alternative expressions of the potential energy in explicit form, using Eqs. (164) and (165); (ii) express the explicit form of Eq. (32) as a sum of two halves; (iii) in one half, replace the variables of integration, $(x_1, x_2, x_3) \leftrightarrow (x_1', x_2', x_3')$, keeping in mind that the integrals are left unchanged; (iv) sum the resulting two halves and compare with the explicit form of Eq. (166).

To get more information on the charge distribution, let us define the tensor potential:

$$\mathcal{V}_{pq}(x_1, x_2, x_3) = G_\chi \int_S \rho_\chi(x_1', x_2', x_3') \times \frac{(x_p - x_p')(x_q - x_q')}{\left[\sum_{s=1}^{3}(x_s - x_s')^2\right]^{1-\chi/2}} \, d^3S' ; \qquad (167)$$

and the self potential-energy tensor:

$$(E_{\rm pot})_{pq} = -\frac{1}{2} \int_S \rho_\chi(x_1, x_2, x_3) \mathcal{V}_{pq}(x_1, x_2, x_3) \, d^3S \quad ; \qquad (168)$$

the above mentioned tensors are manifestly symmetric with respect to the exchange of the indices:

$$\mathcal{V}_{pq}(x_1, x_2, x_3) = \mathcal{V}_{qp}(x_1, x_2, x_3) \quad ; \qquad (169)$$

$$(E_{\rm pot})_{pq} = (E_{\rm pot})_{qp} \quad ; \qquad (170)$$

and the related traces equal their scalar counterparts:

$$\sum_{s=1}^{3} \mathcal{V}_{ss}(x_1, x_2, x_3) = \mathcal{V}(x_1, x_2, x_3) \quad ; \qquad (171)$$

$$\sum_{s=1}^{3} (E_{\rm pot})_{ss} = E_{\rm pot} \quad ; \qquad (172)$$

conform to Eqs. (164), (167), and (166), (168), respectively.

The coincidence of Eqs. (168) and (31) may be verified along the following steps: write the alternative expressions of the potential-energy tensor in explicit form, using Eqs. (167) and (165); (ii) express the explicit form of Eq. (31) as a sum of two halves; (iii) in one half, replace the variables of integration, $(x_1, x_2, x_3) \leftrightarrow (x_1', x_2', x_3')$, keeping in mind that the integrals are left unchanged; (iv) sum the resulting two halves and compare with the explicit form of Eq. (168).

In the special case of gravitational interaction, $\chi = -1$, $G_\chi = G$ (constant of gravitation), $\rho_\chi = \rho$ (mass density), the potential and the potential energy, both in scalar and in tensor form, assume their usual expressions known in literature [C69, Chap. 2, § 10; see also therein a proof for the equivalence of Eqs. (31) and (168) - also illustrative for the equivalence of Eqs. (32) and (166) - where the steps outlined above are followed in a reversed order].

B. An alternative expression of the generalized anisotropy parameters

Let us rewrite Eqs. (107a) and (107b) in a more compact notation, as:

$$(1 - \delta_{3r}) I_{rr} \overline{\Omega}^2 + M\sigma_{rr}^2 + (E_{\rm pot})_{rr} = M\sigma_{rr}^2 - M\zeta_{rr}\sigma^2 \quad ; \qquad r = 1, 2, 3 \quad ; \ (173)$$

and combine Eqs. (105a) and (173) to obtain:

$$\frac{1}{2}\ddot{I}_{rr} = M(\sigma_{rr}^2 - \zeta_{rr}\sigma^2) \quad ; \qquad r = 1, 2, 3 \quad ; \qquad (174)$$

from which the following expression for the generalized anisotropy parameters is derived:

$$\zeta_{rr} = \frac{\sigma_{rr}^2}{\sigma^2} - \frac{1}{2}\frac{\ddot{I}_{rr}}{M\sigma^2} \qquad r = 1, 2, 3 \quad ; \qquad (175)$$

and the related trace, owing to Eqs. (15), (107c), and (107e) reads:

$$\zeta = 1 - \frac{1}{2}\frac{\ddot{I}}{M\sigma^2} \quad ; \qquad r = 1, 2, 3 \quad ; \qquad (176)$$

where ζ exceeds unity for a moment of inertia with respect to the centre of mass decreasing in time, and vice versa.

Finally, the combination of Eqs. (107d) and (175) yields:

$$\widetilde{\sigma}_{rr}^2 = \sigma_{rr}^2 - \frac{1}{2}\frac{\ddot{I}_{rr}}{M} \quad ; \qquad r = 1, 2, 3 \quad ; \qquad (177)$$

which defines the effective velocity dispersion components related to random motions.

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Оригинални научни рад

Циљ рада је стицање бољег увида у три главне поставке које се односе на астрофизичке системе на великим скалама, тј.: (i) формулација тензорских виријалних једначина полазећи од поставки аналитичке механике, (ii) истраживање улоге систематских и случајних кретања у односу на конфигурације у виријалној равнотежи; (iii) степен до кога су систематска и случајна кретања еквивалентна у спљоштавању или издуживању расподеле масе. Тензорске виријалне једначине су формулисане без обзира на природу система и његових конституената генералисањем и проширивањем процедуре коришћене за скаларне виријалне једначине у присуству дискретних под-јединица (Landau and Lifschitz 1966). Нарочито, показано је да је тензор потенцијал-енергија симетричан у односу на замену индекса, $(E_{\rm pot})_{pq} = (E_{\rm pot})_{qp}$. Затим су резултати проширени на континуиране расподеле масе. Детаљно је анализирана улога систематских и случајних кретања у без-сударним, идеалним само-гравитирајућим флуидима укључујући радијалне и тангенцијалне дисперзије брзина у екваторијалној равни, а одговарајућа средња угаона брзина засновaна је као ротација. R3 флуиди су дефинисани као идеални, само-гравитирајући флуиди у виријалној равнотежи са систематском ротацијом око главне осе инерције, рецимо x_3. Одговарајуће виријалне једначине написане су користећи момент тензора инерције, I_{pq}, тензор потенцијал-енергије $(E_{\rm pot})_{pq}$ и генералисани тензор анизотропије, ζ_{pq} (Caimmi and Marmo 2005, Caimmi 2006a). Додатни напор посвећен је истраживању особина осносиметричних и троосних конфигурација. Обједињена теорија систематских и случајних кретања развијена за R3 флуиде узимајући у обзир имагинарну ротацију (Caimmi 1996b, 2006a) и одређене теореме које важе за хомеоидално избраздане Јакобијеве елипсоиде (Caimmi 2006a) проширене су на општији случај R3 флуида. Показано је да је ефекат вишка случајних кретања еквивалентан додатној реалној или имагинарној ротацији, респективно, укључујући спљоштење (дуж екваторијалне равни) или издуживање (дуж ротационе осе). Установљено је да се R3 флуид увек покорава адјунгованој конфигурацији са изотропном расподелом случајних брзина. Такође, додатна ограничења су установљена на степен случајне анизотропије брзина дуж главних оса и то за троосне конфигурације. Формулисан је потребан услов за настанак бифуркационих тачака од осносиметричних до троосних конфигурација у виријалној равнотежи, који је независтан од параметара анизотропије. Урађено је свођење на специјални случај хомеоидално избразданог Јакобијевог елипсоида и неки претходно установљени резултати (Caimmi 2006a) су репродуковани.