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GLOBAL REGULARIZATION METHOD FOR PLANAR RESTRICTED THREE-BODY PROBLEM

M. A. Sharaf[1] and H. R. Dwidar[2]

[1]*Department of Astronomy, Faculty of Science King Abdulaziz University, Jeddah, KSA*
E–mail: *sharaf_adel@hotmail.com*

[2]*Astronomy, Space and Meteorology Science Dept., Faculty of Science, Cairo University, Giza, Egypt*
E–mail: *hanyryd@cu.edu.eg*

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SUMMARY: In this paper, global regularization method for planar restricted three–body problem is purposed by using the transformation $z = x + iy = \nu \cos n(u + iv)$, where $i = \sqrt{-1},\ 0 < \nu \leq 1$ and n is a positive integer. The method is developed analytically and computationally.
For the analytical developments, analytical solutions in power series of the pseudotime τ are obtained for positions and velocities (u, v, u', v') and $(x, y, \dot{x}, \dot{y})$ in both regularized and physical planes respectively, the physical time t is also obtained as power series in τ. Moreover, relations between the coefficients of the power series are obtained for two consequent values of n. Also, we developed analytical solutions in power series form for the inverse problem of finding τ in terms of t. As typical examples, three symbolic expressions for the coefficients of the power series were developed in terms of initial values.
As to the computational developments, the global regularized equations of motion are developed together with their initial values in forms suitable for digital computations using any differential equations solver. On the other hand, for numerical evolutions of power series, an efficient method depending on the continued fraction theory is provided.

Key words. celestial mechanics, Solar system: general

1. INTRODUCTION

The motion of the celestial bodies of the solar system is ruled by Newton's law which states that the attraction between massive bodies is directly proportional to the product of the masses and inversely proportional to the square of their distance. Indeed, a collision between any two objects is marked by the fact that their distance becomes zero, which corresponds to a singularity of Newton's equations. The aim of regularization theory is to transform the singular differential equations into regular ones, thus providing an efficient mathematical tool to analyze motions leading to collisions.

In the planar restricted three-body problem two bodies (called the primaries) revolve around their center of mass in circular orbits under the influence of their mutual gravitational attraction, and a third body attracted by the previous two but not influencing their motion. It is assumed that the third body moves in the plane defined by the two revolving bodies. All the three bodies are considered as point masses. The problem is to find the motion of the third body. The efforts of many famous mathematicians have been devoted to this difficult problem for example Hill (1878) used Jacobi the integral of motion to show that the Earth-Moon distance remains bounded from above for all time (assuming his model for the Sun-Earth-Moon system is valid).

The differential equations of the third body written in rectangular coordinates are an adequate description of its motion when it does not collide with either of the primaries. If a collision takes place, at least one of the two mutual distances between the third body and the primaries vanishes, and the equations of motion are no longer valid; they are singular. Motion near or through collisions can be studied only by using regularized equations of motion.

Regularization of the equations of motion can be achieved by coordinate and time transformations. The time transformation is the essential, the idea is to introduce a new time scale in which the motion slows down near singularity. The simplest among the regularizing transformations was found by (Levi-Civita 1906); it regularizes the collision between two bodies at each eventual recurrence. Such transformations are referred as local regularizations. Recently, (Sharaf and Abouelmagd 2012) developed the equations of motion for photogravitational and oblateness in the elliptic restricted three body problem in terms of the regularized Levi-Civita regularized variables.

There are transformations which can remove both singularities simultaneously. These are called the global regularization (though mathematically they are local operations). These transformations were developed by many pioneers like for example, Thiele (1896), Burrau (1906), Birkhoff (1915), Lemaître (1955), Arenstorf (1963), Deprit and Brouke (1963), Aarseth and Zare (1974) etc. The global mapping proposed by Thiele (1896) has the advantages over other global regularizations in that the equations of motion in this system contain only derivatives and transcendental functions of the dependent variables. In fact the regularization methods become focal points of recent researches (e.g. Waldvogel 2006, Celletti et al. 2011, Jiménez-Pérez and Lacomba 2011, Csillik and Roman 2012) for accurate long term predications of the motion of celestial bodies.

In the absence of a closed analytical solution for a given differential system the power series solution (which is, of course, assumed to be convergent) can serve as the analytical representation of its solution. Moreover, it is worth noting that the power series is one of the most powerful methods of mathematical analysis and is sometimes more convenient than the elementary functions especially when the problems are to be studied numerically on computers. In fact, most computers often use series in the calculations of the majority of elementary functions.

For the numerical solution of ordinary differential equations, either a Runge–Kutta method or an interpolation procedure (Adams, Gauss, etc.) is generally used. Both methods have disadvantages (Fehlberg 1964) in certain situations, however, Runge-Kutta formulae are of a rather low accuracy (truncation errors proportional to h^5, where h is the step size). Rung-Kutta formulae therefore require integration step sizes that could be prohibitively small, resulting in excessive round–off errors and long computation times. On the other hand, interpolation formulae can be of any desired order of accuracy with respect to the truncation error. Interpolation formulae are well suited for problems that can be integrated with constant step size. But if the step size have to be changed, the reconstruction of the difference scheme, which is rather extensive for formulae of higher accuracy, is cumbersome. In such situations, integration by power series expansions is preferable for some types of differential equations. Estes and Lancaster (1970) developed recurrence formulae for coefficients in power series expansions of the solution of the planar restricted three body problem in the regularized form of Thiele. Sharaf et al. (2012) developed explicit analytical solutions in power series forms for the restricted three-body problem in three dimensional space.

In the present paper, a global regularization method for planar restricted three–body problem is purposed by using the transformation $z = x + iy = \nu \cos n(u + iv)$, $i = \sqrt{-1}$, $0 < \nu \le 1$ and n positive integer. The method is developed analytically and computationally. For analytical developments, explicit analytical solutions in power series of the pseudo-time τ are obtained for positions and velocities (u, v, u', v') and $(x, y, \dot{x}, \dot{y})$ in both regularized and physical planes respectively, the physical time t is also obtained as power series in τ. Moreover, relations between the coefficients of the power series are obtained for two consequent values of n. Also, we developed explicite analytical solutions in power series form for the inverse problem to find τ in terms of t. As typical examples, three symbolic expressions for the coefficients of the power series were developed in terms of initial values. The importance of analytical power series representations is that they are invariant under many operations because the addition, multiplication, exponentiation, integration, differentiation etc. of power series is also a power series. A fact which provides excellent flexibility in dealing with analytical as well as computational developments for problems relating to the motion of the third body. In addition, the series solution enables us to obtain full numerical solutions for any given set of initial values.

While for the computational developments of the present method, the global regularized equations of motion are developed together with their initial values in forms suitable for digital computations using any differential equations solver. On the other hand, for numerical evolutions of power series, an efficient method depending on the continued fraction theory is provided.

2. BASIC FORMULATIONS

2.1. The coordinate system

In the present paper we shall adopt new coordinate system with the origin at distance ν , $0 < \nu \le 1$, from the more massive primary and its x-axis passing through the primaries (Fig. 1).

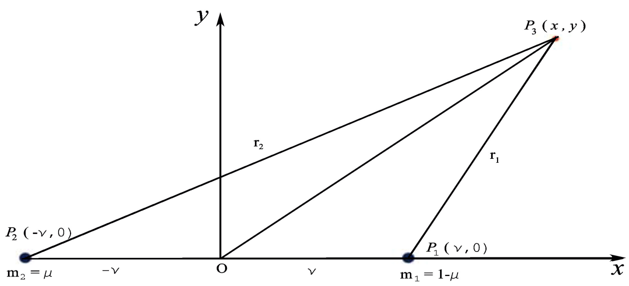

Fig. 1. *Configuration of the restricted three body problem in the new coordinate system.*

So, the primaries P_1 and P_2 are located symmetrically with respect to the origin at $(\pm\,\nu,\ 0)$ and r_1 , r_2 represent distances between the primaries and the infinitesimal third body $P_3(x, y)$ and are given as:

$$r_1^2 = (x-\nu)^2 + y^2, \tag{1}$$

$$r_2^2 = (x+\nu)^2 + y^2. \tag{2}$$

2.2. Regularizing transformation

We introduce the general global regularizing transformation:

$$x = \nu \cos\ nu\ \cosh nv, \tag{3}$$

$$y = -\nu\ sin\ nu\ \sinh\ nv\,, \tag{4}$$

and n is a positive integer. The transformation is general in the sense that it includes the well–known Thiele-Burrau regularizing transformation (introduced by Thiele (1896) and generalized by Burrau (1906)) when $n = 1$ and $\nu = 1/2$ and Broucke's regularizing transformation (Broucke 1965) when n is any real number and $\nu = 1/2$. The lines $nu = \text{const}$ are hyperbolas and the lines $nv = \text{const}$ are ellipses in the x, y plane, since:

$$\frac{x^2}{\cosh^2 nv} + \frac{y^2}{\sinh^2 nv} = \nu^2 \text{ and } \frac{x^2}{\cos^2 nu} - \frac{y^2}{\sin^2 nu} = \nu^2.$$

The center of the confocal hyperbolas and ellipses is at the origin and the focuses are at the primaries. There are two infinite sets of values of u, v for each pair x, y as far as n is not an integer nor rational. If $\tilde{u}, \tilde{v}$ is a couple corresponding to x, y, then all such couples are

$$\left(\tilde{u} + \frac{2k\pi}{\text{n}}, \tilde{v}\right) \text{ and } \left(-\tilde{u} + \frac{2k\pi}{\text{n}}, -\tilde{v}\right)\ ; k = 0, 1, 2, \cdots .$$

From Eqs. (1), (2), (3) and (4) we get:

$$r_1 = \nu\ (\cosh nv - \cos nu), \tag{5}$$

$$r_2 = \nu\ (\cosh nv + \cos nu). \tag{6}$$

2.3. Equations of motion

The equations of motion for the planar restricted three–body problem of non dimensional variables x, y and time t in the above coordinate system while rotating with the mean motion are:

$$\ddot{x} - 2\dot{y} = \frac{\partial \Phi}{\partial x}, \tag{7a}$$

$$\ddot{y} + 2\dot{x} = \frac{\partial \Phi}{\partial y}, \tag{7b}$$

where:

$$\Phi = \frac{1}{2}[(1-\mu)r_1^2 + \mu r_2^2] + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}\,, \tag{8}$$

and μ is the ratio of the mass of the smaller primary to the sum of two primaries. Here, a dot over a symbol denotes the derivative with respect to t. The Jacobi constant of motion is:

$$J = 2\Phi - (\dot{x}^2 + \dot{y}^2).$$

3. REGULARIZED EQUATIONS OF MOTION

3.1. Analytical developments

By the classical technique for obtaining the regularizing equations of motion (e.g. Szebehely 1967) we get:

$$n^2u'' - 2n^3\nu^2v'(\cosh^2 nv - \cos^2 nu) = \frac{\partial \Phi^*}{\partial u}, \tag{9a}$$

$$n^2v'' + 2n^3\nu^2u'(\cosh^2 nv - \cos^2 nu) = \frac{\partial \Phi^*}{\partial v}, \tag{9b}$$

$$\frac{dt}{d\tau} = r_1r_2 = \nu^2(\cosh^2 nv - \cos^2 nu), \tag{9c}$$

where the prime accents indicate differentiation with respect to a pseudo-time τ and:

$$\Phi^* = n^2r_1r_2(\Phi - \frac{J}{2}), \tag{10}$$

therefore both singularities ($r_1 \to 0, r_2 \to 0$) are removed simultaneously when Φ is multiplied by r_1r_2 and for this, the equations of motion Eq. (9) are the global regularized equations of motion (GRE) associated with the general coordinate transformations of Eqs. (3) and (4) with Φ^* given as:

$$\Phi^* = \frac{1}{2}n^2\nu\,[\cos nu\,\{2-4\mu+J\nu\cos nu-$$
$$-\nu^3\cos^3 nu\} + \cosh nv\,\{2+2(1-2\mu)\nu^3\cos^3 nu+$$
$$+\,\nu\cosh nv\,(-J+\,\nu^2\cosh nv\,(2(2\mu-1)\cos nu+$$
$$+\cosh nv)\,)\,\}\,]$$

from which we get:

$$\frac{\partial\Phi^*}{\partial u} = n^3\nu\,(2\mu-1-J\nu\cos nu+2\nu^3\cos^3 nu-$$
$$-\,(2\mu-1)\nu^3\cosh nv\,\left(-3\cos^2 nu\,+\right.$$
$$\left.+\cosh^2 nv\,\right))\sin nu\,,$$
$$\frac{\partial\Phi^*}{\partial v} = -n^3\nu\,\left(-1+(2\mu-1)\nu^3\cos^3 nu+\right.$$
$$+\,\nu\cosh nv\,(J+\;+\nu^2\,(3(1-2\mu)\cos nu-$$
$$\left.-2\cosh nv\,)\,\cosh nv\,)\,\right)\sinh nv\,, \tag{11}$$

also:

$$u'^2+v'^2 = n^2 r_1 r_2(\dot{x}^2+\dot{y}^2).$$

3.2. Computational developments

For the computational purposes, the equations of motion Eq. (9) are most conveniently given as first order system:

$$n^2 u' = \xi\,, \tag{12a}$$
$$n^2 v' = \eta\,, \tag{12b}$$
$$\xi' = 2\,n\,\nu^2\eta\,(\,\cosh^2 nv-\cos^2 nu)+\frac{\partial\Phi^*}{\partial u}, \tag{12c}$$
$$\eta' = -2\,n\,\nu^2\,\xi\,(\,\cosh^2 nv-\cos^2 nu)+\frac{\partial\Phi^*}{\partial v}, \tag{12d}$$
$$t' = r_1 r_2 = \nu^2(\cosh^2 nv-\cos^2 nu), \tag{12e}$$

where $\dfrac{\partial\Phi^*}{\partial u}$ and $\dfrac{\partial\Phi^*}{\partial v}$ are given by Eqs. (11). Eqs. (12) are to be solved subject to the initial conditions that:

$$\text{at}: \quad \tau=\tau_0\,,\; u=u_0\,,\; v=v_0\,;\; u'=u'_0\,,\; v'=v'_0, \tag{13}$$

which could be computed for given values of n and ν from the known initial conditions in the physical plane:

$$\text{at} \quad t=t_0\,,\; x=x_0\,,\; y=y_0\,;\; \dot{x}=\dot{x}_0\,,\; y=\dot{y}_0, \tag{14}$$

as follows. From Eqs. (1) and (2) we have

$$r_{10} = \left((x_0-\nu)^2+y_0^2\right)^{1/2}, \tag{15a}$$
$$r_{20} = \left((x_0+\nu)^2+y_0^2\right)^{1/2}, \tag{15b}$$

then from Eqs. (5) and (6) we get for v_0 and u_0 the expressions:

$$v_0 = \frac{1}{n}\ln\left[\frac{1}{2\nu}\left\{\,r_{10}+r_{20}+\sqrt{(r_{10}+r_{20})^2-4\nu^2}\,\right\}\right], \tag{16a}$$
$$u_0 = \frac{1}{n}\cos^{-1}\left(\frac{r_{20}-r_{10}}{2\nu}\right). \tag{16b}$$

Differentiating Eqs. (1) and (2) with respect to t and then applying Eq. (12e) we get:

$$\dot{x} = \frac{\nu}{n\,r_1 r_2}\,[v'\cos nu\sinh\,nv\,-u'\sin\,nu\,\cosh\,nv\,]\,,$$
$$\dot{y} = \frac{-\nu}{n\,r_1 r_2}\,[v'\sin nu\cosh\,nv\,+u'\cos\,nu\,\sinh\,nv\,]\,.$$

Solving these two equations for u'_0, v'_0 in terms of $\dot{x}_0,\ \dot{y}_0,\ u_0$ and v_0 we obtain:

$$u'_0 = -\nu n\,(\dot{x}_0\cosh nv_0\sin nu_0-\dot{y}_0\cos nu_0\sinh nv_0)\,, \tag{16c}$$
$$v'_0 = -\nu n\,(\dot{y}_0\cosh nv_0\sin nu_0+\dot{x}_0\cos nu_0\sinh nv_0)\,. \tag{16d}$$

Finally, the Jacobi constant J is computed from:

$$J = (1-\mu)r_{10}^2+\mu r_{20}^2+\frac{2(1-\mu)}{r_{10}}+\frac{2\mu}{r_{20}}-(\dot{x}_0^2+\dot{y}_0^2). \tag{16e}$$

Eqs. (16) are the required initial conditions for the equations of motion Eq. (12). Finally, we have from Eqs. (5) and (6) the following useful formulae:

$$\begin{aligned}
r_1 &= 2\nu\left(\sinh^2\frac{nv}{2}+\sin^2\frac{nu}{2}\right),\\
r_2 &= 2\nu\left(\sinh^2\frac{nv}{2}+\cos^2\frac{nu}{2}\right),\\
2\sinh^2\frac{nv}{2} &= \frac{1}{2\nu}\;(r_1+r_2-2\nu)\,,\\
2\cosh^2\frac{nv}{2} &= \frac{1}{2\nu}\;(r_1+r_2+2\nu)\,,\\
2\sin^2\frac{nu}{2} &= \frac{1}{2\nu}\;(r_1-r_2+2\nu)\,,\\
2\cos^2\frac{nu}{2} &= \frac{1}{2\nu}\;(r_2-r_1+2\nu)\,.
\end{aligned} \tag{17}$$

4. POWER SERIES SOLUTIONS

4.1. Basic recurrent equations

The first step in solving the equations of motion by power series, is to put them as a system of the *second degree* in the sense that its final form depends on the products of two dependent variables only. It is known for a long time (Steffenesen 1957) that this form of equations is particularly well prepared for the substitution of power series and the identification of equal powers of the independent variable (τ in our case). In this respect, Eqs. (9) with Eqs. (11) are written as:

$$\alpha'^{(n)} - 2n^2 w^{(n)}\beta^{(n)} = a^{(n)} f^{(n)}, \tag{18a}$$

$$\beta'^{(n)} + 2n^2 w^{(n)}\alpha^{(n)} = c^{(n)} g^{(n)}, \tag{18b}$$

$$a^{(n)} = \sin nu\ ,\ b^{(n)} = \cos nu\ ,$$

$$c^{(n)} = \sinh nv\ ,\ d^{(n)} = \cosh nv\ , \tag{18c}$$

$$q^{(n)} = d^{(n)} \times d^{(n)}\ ,\ p^{(n)} = b^{(n)} \times b^{(n)}, \tag{18d}$$

$$u' = \alpha^{(n)}\ ,\ v' = \beta^{(n)}\ ,\ t' = w^{(n)} = \nu^2(q^{(n)} - p^{(n)}), \tag{18e}$$

$$a'^{(n)} = n\alpha^{(n)} b^{(n)}\ ,\quad b'^{(n)} = -n\alpha^{(n)} a^{(n)}\ ,$$

$$c'^{(n)} = n\beta^{(n)} d^{(n)}\ ,\quad d'^{(n)} = n\beta^{(n)} c^{(n)}, \tag{18f}$$

$$f^{(n)} = k_1 + k_2 b^{(n)} + r^{(n)} p^{(n)} + k_4 d^{(n)} q^{(n)}, \tag{18g}$$

$$r^{(n)} = k_3 b^{(n)} - 3k_4 d^{(n)}, \tag{18h}$$

$$g^{(n)} = k_5 - 2k_4 b^{(n)} q^{(n)} + k_2 d^{(n)} + k_3 d^{(n)} q^{(n)}, \tag{18i}$$

$$k_1 = (2\mu - 1)n\nu\quad ;\quad k_2 = -Jn\nu^2\quad ;$$
$$k_3 = 2n\nu^4\ ; k_4 = -(2\mu - 1)n\nu^4\quad ;\quad k_5 = n\nu. \tag{19}$$

The second step in the expansion is to assume the power series of the variables as:

$$u = \sum_{i=0}^{\infty} u_i^{(n)}\tau^i,\quad v = \sum_{i=0}^{\infty} v_i^{(n)}\tau^i,\quad t = \sum_{i=0}^{\infty} t_i^{(n)}\tau^i,$$
$$a^{(n)} = \sum_{i=0}^{\infty} a_i^{(n)}\tau^i,\quad b^{(n)} = \sum_{i=0}^{\infty} b_i^{(n)}\tau^i,\quad c^{(n)} = \sum_{i=0}^{\infty} c_i^{(n)}\tau^i,$$
$$p^{(n)} = \sum_{i=0}^{\infty} p_i^{(n)}\tau^i,\quad q^{(n)} = \sum_{i=0}^{\infty} q_i^{(n)}\tau^i,\quad w^{(n)} = \sum_{i=0}^{\infty} w_i^{(n)}\tau^i,$$
$$r^{(n)} = \sum_{i=0}^{\infty} r_i^{(n)}\tau^i,\quad \beta^{(n)} = \sum_{i=0}^{\infty} \beta_i^{(n)}\tau^i,\quad \alpha^{(n)} = \sum_{i=0}^{\infty} \alpha_i^{(n)}\tau^i,$$
$$f^{(n)} = \sum_{i=0}^{\infty} f_i^{(n)}\tau^i,\quad g^{(n)} = \sum_{i=0}^{\infty} g_i^{(n)}\tau^i. \tag{20}$$

The third step in the expansion is to substitute the above series into Eqs. (18) and equating the coefficients of τ^i on both sides of each equation. In doing so we used two simple rules:

(i) If $X = \sum_{i=0}^{\infty} x_i\tau^i$, then $X' = \sum_{i=0}^{\infty}(i+1)x_{i+1}\tau^i$,

(ii) If $Y = \sum_{i=0}^{\infty} y_i\tau^i$, then $X \times Y = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} x_j y_{i-j}\right)\tau^i$

and we get:

$$\alpha'^{(n)} = \sum_{i=0}^{\infty}(i+1)\alpha_{i+1}^{(n)}\tau^i, \tag{21a}$$

$$\beta_i^{(n)} = (i+1)v_{i+1}^{(n)}, \tag{21b}$$

$$w^{(n)}\beta^{(n)} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} w_j^{(n)}\beta_{i-j}^{(n)}\right)\tau^i, \tag{21c}$$

$$a^{(n)} f^{(n)} = \sum_{i=0}^{\infty}\left(\sum_{j=0}^{i} a_j^{(n)} f_{i-j}^{(n)}\right)\tau^i. \tag{21d}$$

Substituting Eqs. (21) into Eq. (18a) and then equating the coefficients of τ^i in both sides of the equation we get:

$$(i+1)\alpha_{i+1}^{(n)} = 2n^2\sum_{j=0}^{i} w_j^{(n)}\beta_{i-j}^{(n)} + \sum_{j=0}^{i} a_j^{(n)} f_{i-j}^{(n)}. \tag{22}$$

Similarly, we get from Eq. (18b) that:

$$(i+1)\beta_{i+1}^{(n)} = \sum_{j=0}^{i} c_j^{(n)} g_{i-j}^{(n)} - 2n^2\sum_{j=0}^{i} w_j^{(n)}\alpha_{i-j}^{(n)}. \tag{23}$$

Also we get from Eqs. (18g), (18h) and (18i) that:

$$f_i^{(n)} = k_1\delta_i + k_2 b_i^{(n)} + \sum_{j=0}^{i} r_j^{(n)} p_{i-j}^{(n)} + k_4\sum_{j=0}^{i} d_j^{(n)} q_{i-j}^{(n)}, \tag{24a}$$

$$r_i^{(n)} = k_3 b_i^{(n)} - 3k_4 d_i^{(n)}, \tag{24b}$$

$$g_i^{(n)} = k_5\delta_i - 2k_4\sum_{j=0}^{i} b_j^{(n)} q_{i-j}^{(n)} + k_2 d_i^{(n)} + k_3\sum_{j=0}^{i} d_j^{(n)} q_{i-j}^{(n)}, \tag{25}$$

where δ_i is the Kronecker symbol which equals one if i is zero, and zero otherwise.

To complete the recursive computations we need to set up some relations between the coefficients of the series as follows:

$$p_i^{(n)} = \sum_{j=0}^{i} b_j^{(n)} b_{i-j}^{(n)} \ , \ q_i^{(n)} = \sum_{j=0}^{i} d_j^{(n)} d_{i-j}^{(n)} \ , \qquad w_i^{(n)} = \nu^2 (q_i^{(n)} - p_i^{(n)}), \tag{26}$$

$$\alpha_i^{(n)} = (i+1) u_{i+1}^{(n)} \quad , \quad w_i^{(n)} = (i+1) t_{i+1}^{(n)} \ , \tag{27}$$

$$(i+1) a_{i+1}^{(n)} = n \sum_{j=0}^{i} b_j^{(n)} \alpha_{i-j}^{(n)} \ , \qquad (i+1) b_{i+1}^{(n)} = -n \sum_{j=0}^{i} a_j^{(n)} \alpha_{i-j}^{(n)}, \tag{28}$$

$$(i+1) c_{i+1}^{(n)} = n \sum_{j=0}^{i} d_j^{(n)} \beta_{i-j}^{(n)} \ , \qquad (i+1) d_{i+1}^{(n)} = n \sum_{j=0}^{i} c_j^{(n)} \beta_{i-j}^{(n)} \ . \tag{29}$$

Note that advantage may be taken of the ij- symmetry in the first and second equation of Eq. (26) to nearly halve the number of arithmetic operations for those recursions.

4.2. Evaluation of the power series

In fact, continued fraction expansions are generally far more efficient tools to evaluate classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than in the case of series. Due to the importance of accurate evaluations and the efficiency of continued fractions, we propose to use them as the computational tools for numerical evalution. To do so, two steps are to be performed:

1. Transformation of the given power series into continued fraction (Subsection 4.2.1).
2. Evaluation of the resulting continued fraction (Subsection 4.2.2).

4.2.1. Euler's transformation

Generally an infinite series (a power series is special case of it) of functions could be converted into a continued fraction according to Euler transformation (Battin 1999) which is:

$$\sum_{k=0}^{\infty} U_K \equiv \cfrac{N_1}{D_1 + \cfrac{N_2}{D_2 + \cfrac{N_3}{D_3 + \cfrac{N_4}{...}}}}$$

$$\equiv \frac{N_1}{D_1+} \frac{N_2}{D_2+} \frac{N_3}{D_3+} \frac{N_4}{D_4+} + ...$$

where:

$$N_1 = U_0 \ ; \ \ N_2 = U_1; \ \ N_i = -U_{i-1} \times U_{i-3}, \quad \forall i \geq 3,$$

$$D_1 = 1 \ \ ; \ \ D_j = U_{j-2} + U_{j-1}, \quad \forall j \geq 2.$$

4.2.2. Top-Down continued fraction evaluation

There are several methods available for the evaluation of continued fraction. Traditionally, the fraction was either computed from the bottom up, or the numerator and denominator of the n-th convergent were accumulated separately with three-term recurrence formulae. The drawback of the first method is obviously the decision that has to be made on how far down the fraction one has to go in order to ensure desired convergence. The draw back to the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top down while avoiding numerical difficulties would be ideal from a programming standpoint.

Gautschi (1967) proposed a very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as:

$$\Omega \ = \ \frac{N_1}{D_1+} \ \ \frac{N_2}{D_2+} \ \ \frac{N_3}{D_3+} \ \ ...$$

then initialize the following parameters:

$$A_1 = 1 \ \ , \ \ \ B_1 = N_1/D_1 \ , \Omega_1 = N_1/D_1$$

and iterate (k=1,2,. . .) according to:

$$A_{k+1} = \cfrac{1}{1 + \cfrac{N_{k+1}}{D_k D_{k+1}} A_k},$$

$$B_{k+1} = \ (A_k - 1) \, B_k \, ,$$

$$\Omega_{k+1} = \ \Omega_k + B_{k+1} \ .$$

In the limit, the Ω sequence converges to the value of the continued fraction.

4.3. Utilization

When the pseudo-time τ is sufficiently large, we may (as usually done for all initial value problems divide the interval of τ into some intervals each of short length e.g. $[\tau - \tau_0]$ may be divided into intervals: $[\tau_1 - \tau_0], [\tau_2 - \tau_1], ...[\tau - \tau_q]$ such that $\tau_h - \tau_{h-1} \,!\, \tau - \tau_0$. Then solve the initial value problem for the first interval to find the solution at the value τ_1. The solution at τ_1 could then be used as the initial condition for the second interval and so on. By this artifice one usually needs a small number of the coefficients for the power series representation in each interval.

5. SYMBOLIC DEVELOPMENTS

5.1. Symbolic algorithm

Symbolic expressions for coefficients of the power series representation of GRE, could be obtained as follows:
For given values of n and ν, the initial conditions u_0, v_0, α_0, β_0, at τ_0 can be computed as illustrated in Section 3.2, Eqs. (18c) can be solved for $a_0^{(n)}$, $b_0^{(n)}$, $c_0^{(n)}$, $d_0^{(n)}$, and the higher order coefficients ($i \geq 1$) can then be computed in full recursive manner from the above equations of Subsection 3.1.

Examples of the symbolic expressions $u_i^{(n)}, v_i^{(n)}$ and $t_i^{(n)}$; $i = 1, 2$ and 3 are listed in Table 1 for any values of n and ν. To simplify the notations the upper subscripts (n) on the coefficients will be omitted here (also in all the other tables). Also note that $\xi_0 = \alpha_0 = u_1$ and $\eta_0 = \beta_0 = v_1$.

Table 1: Symbolic expressions of $u_i^{(n)}$, $v_i^{(n)}$ and $t_i^{(n)}$; $i = 1, 2, 3.$ for any values n and ν

$$
\begin{aligned}
u_1 &= \xi_0, \\
u_2 &= -\frac{1}{2} n\nu \left(a_0 \left(J\nu b_0 - 2b_0^3 + 3\left(1-2\mu\right)\nu^3 b_0^2 d_0 + \right.\right. \\
&\quad \left.\left. + \left(2\mu-1\right)\left(\nu^3 d_0^3 - 1\right)\right) + 2n\nu(b_0^2 - d_0^2)\eta_0\right), \\
u_3 &= \frac{1}{6} n^2 \nu \left(\nu c_0 d_0 \left(2n\nu d_0 \left(1 - J\nu d_0 + 2d_0^3\right) + 3\left(1 - \right.\right.\right. \\
&\quad \left. -2\mu\right)\nu^2 a_0 d_0 \eta_0 + 4n\eta_0^2\right) - (2\mu-1)\,\nu^3 b_0^3 d_0 \left(4n\nu^2 c_0 d_0 - \right. \\
&\quad \left. -3\xi_0\right) + 2\left(1 - 2n^2\nu^3\right) b_0^4 \xi_0 + \nu\left(Ja_0^2 - 4n^2\nu^2 d_0^4\right)\xi_0 + \\
&\quad + b_0^2 \left(\nu^2 c_0 \left(2n\left(J\nu d_0 - 2d_0^3 - 1\right) + 3\left(2\mu-1\right)\nu a_0 \eta_0\right) - \right. \\
&\quad \left. - \left(J\nu + 6a_0^2 - 8n^2\nu^3 d_0^2\right)\xi_0\right) + b_0 (4n(2\mu-1)\nu^5 c_0 d_0^4 + \\
&\quad + (-(-1+2\mu)(-1+\nu^3 d_0(6a_0^2 + d_0^2)) + 4n\nu a_0 \eta_0)\xi_0)\left.\right), \\
v_1 &= \eta_0, \\
v_2 &= \frac{1}{2} n\nu (c_0 (1 + d_0(-J\nu + 2d_0((2\mu-1)\nu^3 b_0 + d_0))) + \\
&\quad + 2n\nu(b_0^2 - d_0^2)\xi_0), \\
v_3 &= -\frac{1}{6} n^2 \nu \left(\eta_0 (4n^2\nu^3 b_0^4 - 8n^2\nu^3 b_0^2 d_0^2 + c_0^2 (J\nu - 6d_0^2) - \right. \\
&\quad -2(2\mu-1)\nu^3 b_0 d_0 (2c_0^2 + d_0^2) + d_0(-1 + J\nu d_0 + \\
&\quad + (-2 + 4n^2\nu^3) d_0^3) + 4n\nu c_0 d_0 \xi_0) + 2\nu a_0 (-n\nu (b_0^2 - \\
&\quad - d_0^2)(-1 + 2\mu - J\nu b_0 + 2b_0^3 + 3(2\mu-1)\nu^3 b_0^2 d_0 + \\
&\quad \left. + (1-2\mu)\nu^3 d_0^3) + (2\mu-1)\nu^2 c_0 d_0^2 \xi_0 + 2n b_0 \xi_0^2)\right), \\
t_1 &= \nu^2 \left(d_0^2 - b_0^2\right), \\
t_2 &= n\nu^2 \left(c_0 d_0 \eta_0 + a_0 b_0 \xi_0\right), \\
t_3 &= \frac{1}{3} n^2 \nu^2 \left(2n\nu^2 a_0 b_0 (d_0^2 - b_0^2)\eta_0 + d_0^2 \eta_0^2 + c_0^2 (\nu d_0 (1 + \right. \\
&\quad + d_0(-J\nu + +2d_0((-1+2\mu)\nu^3 b_0 + d_0))) + \eta_0^2) + \\
&\quad + 2n\nu^2 c_0 d_0 (b_0^2 - d_0^2)\xi_0 + b_0^2 \xi_0^2 + a_0^2 (\nu b_0 (2\mu - 1 - J\nu b_0 + \\
&\quad \left. + 2b_0^3 + 3(-1+2\mu)\nu^3 b_0^2 d_0 + (1-2\mu)\nu^3 d_0^3) - \xi_0^2)\right).
\end{aligned}
$$

5.2. Power series of $x, y, \dot{x}$ and $\dot{y}$

Eqs. (3) and (4), could be written in power series forms as

$$\sum_{i=0}^{\infty} x_i^{(n)} \tau^i = \nu \sum_{i=0}^{\infty} b_i^{(n)} \tau^i \times \sum_{i=0}^{\infty} d_i^{(n)} \tau^i \,,$$

$$\sum_{i=0}^{\infty} y_i^{(n)} \tau^i = -\nu \sum_{i=0}^{\infty} a_i^{(n)} \tau^i \times \sum_{i=0}^{\infty} c_i^{(n)} \tau^i \,,$$

then

$$x_i^{(n)} = \nu \sum_{j=0}^{i} b_j^{(n)} d_{i-j}^{(n)} \quad ; \quad y_i^{(n)} = -\nu \sum_{j=0}^{i} a_j^{(n)} c_{i-j}^{(n)} \,. \tag{30}$$

From Section 3.2 we have:

$$\dot{x} = \frac{v' \, b^{(n)} c^{(n)} - u' a^{(n)} d^{(n)}}{n\nu \left(q^{(n)} - p^{(n)}\right)}, \tag{31a}$$

$$\dot{y} = \frac{v' \, a^{(n)} d^{(n)} + u' b^{(n)} c^{(n)}}{-n\nu \left(q^{(n)} - p^{(n)}\right)}. \tag{31b}$$

Let the power series representations of $\dot{x}$ and ẏ are

$$\dot{x} = \sum_{j=0}^{\infty} R_j^{(n)} \tau^j, \tag{32a}$$

$$\dot{y} = \sum_{j=0}^{\infty} T_j^{(n)} \tau^j. \tag{32b}$$

Using the power series for each function of the numerators and denominators of Eqs. (31), and the multiplication rule for power series as mentioned above, each of the equations reduces to the form:

$$\frac{\sum_{i=0}^{\infty} X_i \tau^i}{\sum_{i=0}^{\infty} Y_i \tau^i} = \sum_{j=0}^{\infty} Z_j \tau^j \ ,$$

where the Z's coefficients computed recursively from:

$$Z_j = \frac{1}{Y_0}\left(X_j - \sum_{k=1}^{j} Y_k \ Z_{j-k}\right) \ .$$

Consequently, we get for the R's and T's coefficients of Eqs. (32) the expressions:

$$\begin{aligned} R_j^{(n)} &= \frac{1}{G_0^{(n)}}\left(F_j^{(n)} - \varepsilon_j \sum_{k=1}^{j} G_k^{(n)} R_{j-k}^{(n)}\right), \\ T_j^{(n)} &= \frac{-1}{G_0^{(n)}}\left(Q_j^{(n)} + \varepsilon_j \sum_{k=1}^{j} G_k^{(n)} T_{j-k}^{(n)}\right), \end{aligned} \qquad (33)$$

where ε_j equals zero if j is zero, and one otherwise, and:

$$Q_i^{(n)} = S_i^{(n)} + E_i^{(n)} \quad , \quad F_i^{(n)} = P_i^{(n)} - H_i^{(n)} \quad ,$$

$$P_i^{(n)} = \sum_{j=0}^{i}(i-j+1)\beta_{i-j+1}^{(n)} \sum_{k=0}^{j} b_k^{(n)} c_{j-k}^{(n)} \quad ,$$

$$S_i^{(n)} = \sum_{j=0}^{i}(i-j+1)\beta_{i-j+1}^{(n)} \sum_{k=0}^{j} a_k^{(n)} d_{j-k}^{(n)} \quad ,$$

$$H_i^{(n)} = \sum_{j=0}^{i}(i-j+1)\alpha_{i-j+1}^{(n)} \sum_{k=0}^{j} a_k^{(n)} d_{j-k}^{(n)} \quad ,$$

$$E_i^{(n)} = \sum_{j=0}^{i}(i-j+1) \ \alpha_{i-j+1}^{(n)} \sum_{k=0}^{j} b_k^{(n)} c_{j-k}^{(n)} \quad ,$$

$$G_i^{(n)} = n\nu \sum_{j=0}^{i}(d_j^{(n)} d_{i-j}^{(n)} - b_j^{(n)} b_{i-j}^{(n)}).$$

By successive use of the first relation of Eq. (33) we get:

$$R_j^{(n)} = \frac{(-1)^j}{(G_0^{(n)})^{j+1}} \times$$

$$\times \begin{vmatrix} G_1^{(n)} F_0^{(n)} & -G_0^{(n)} F_1^{(n)} & G_0^{(n)} & 0 & \cdots & 0 \\ G_2^{(n)} F_0^{(n)} & -G_0^{(n)} F_2^{(n)} & G_1^{(n)} & G_0^{(n)} & \cdots & 0 \\ G_3^{(n)} F_0^{(n)} & -G_0^{(n)} F_3^{(n)} & G_2^{(n)} & G_1^{(n)} & \cdots & 0 \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots\cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots \\ G_{j-1}^{(n)} F_0^{(n)} & -G_0^{(n)} F_{j-1}^{(n)} & G_{j-2}^{(n)} & G_{j-3}^{(n)} & \cdots & G_0^{(n)} \\ G_j^{(n)} F_0^{(n)} & -G_0^{(n)} F_j^{(n)} & G_{j-1}^{(n)} & G_{j-2}^{(n)} & \cdots & G_1^{(n)} \end{vmatrix}$$

Similarly $T_j^{(n)}$ can be obtained in a determinant form by a successive use of the second relation in Eq. (33). Examples of the symbolic expressions $x_i^{(n)}$, $y_i^{(n)}$, $R_i^{(n)}$ and $T_i^{(n)}$; $i = 0, 1, 2.$ are listed in Table 2 for any values of n and ν.

Table 2: Symbolic expressions of $x_i^{(n)}$, $y_i^{(n)}$, $R_i^{(n)}$ and $T_i^{(n)}$, $i = 0, 1, 2$ for any values n and ν

$x_0 = \nu b_0 d_0$, $\quad y_0 = -\nu a_0 c_0$,

$x_1 = \nu(b_1 d_0 + b_0 d_1)$, $\quad y_1 = -\nu(a_1 c_0 + a_0 c_1)$,

$y_2 = -\nu(a_2 c_0 + a_1 c_1 + a_0 c_2)$, $\quad x_2 = \nu(b_2 d_0 + b_1 d_1 + b_0 d_2)$,

$R_0 = (b_0 c_0 \beta_1 - a_0 d_0 \alpha_1)/\lambda_2$,

$$\begin{aligned} R_1 &= (-(a_1 d_0 + a_0 d_1)\alpha_1 - 2a_0 d_0 \alpha_2 + (b_1 c_0 + b_0 c_1)\beta_1 - \\ &\quad -\lambda_3)/\lambda_2, \\ R_2 &= \frac{1}{\lambda_2}(-(a_2 d_0 + a_1 d_1 + a_0 d_2)\alpha_1 - 2(a_1 d_0 + a_0 d_1)\alpha_2 - \\ &\quad -3a_0 d_0 \alpha_3 + (b_2 c_0 + b_1 c_1 + b_0 c_2)\beta_1 + ((b_1^2 + 2b_0 b_2 - \\ &\quad -d_1^2 - 2d_0 d_2)(b_0 c_0 \beta_1 - a_0 d_0 \alpha_1))/\lambda_1 + 2(b_1 c_0 + \\ &\quad +b_0 c_1)\beta_2 - ((2d_0 d_1 - 2b_0 b_1)(-(a_1 d_0 + a_0 d_1)\alpha_1 - \\ &\quad -2a_0 d_0 \alpha_2 + (b_1 c_0 + b_0 c_1)\beta_1 - \lambda_3))/\lambda_1 + 3b_0 c_0 \ \beta_3), \\ T_0 &= (a_0 d_0 \beta_1 - b_0 c_0 \alpha_1)/\lambda_2, \\ T_1 &= (-(b_1 c_0 + b_0 c_1)\alpha_1 + 2b_0 c_0 \alpha_2 + (a_1 d_0 + a_0 d_1)\beta_1 - \\ &\quad -\lambda_4)/\lambda_2, \\ T_2 &= \frac{1}{\lambda_2}((b_2 c_0 + b_1 c_1 + b_0 c_2)\alpha_1 + 2(b_1 c_0 + b_0 c_1)\alpha_2 + \\ &\quad +3b_0 c_0 \alpha_3 + (a_2 d_0 + a_1 d_1 + a_0 d_2)\beta_1 - \lambda_4 + 2(a_1 d_0 + \\ &\quad +a_0 d_1)\beta_2 - ((2d_0 d_1 - 2b_0 b_1)((b_1 c_0 + b_0 c_1)\alpha_1 + \\ &\quad +2b_0 c_0 \alpha_2 + (a_1 d_0 + a_0 d_1)\beta_1 - \lambda_4))/\lambda_1 + 3a_0 d_0 \beta_3), \end{aligned}$$

where:

$\lambda_1 = d_0^2 - b_0^2 \quad , \quad \lambda_2 = n\nu\lambda_1$,

$\lambda_3 = ((2d_0 d_1 - 2b_0 b_1)(b_0 c_0 \beta_1 - a_0 d_0 \alpha_1))/\lambda_1 + 2b_0 c_0 \beta_2$,

$\lambda_4 = ((2d_0 d_1 - 2b_0 b_1)(b_0 c_0 \alpha_1 + a_0 d_0 \beta_1))/\lambda_1 + 2a_0 d_0 \beta_2$.

5.3. Relations between the coefficients for two consequent values of n

Since:

$$\sin(n+1)u = \sin nu \cos u + \cos nu \sin u,$$

using the power series for the functions on both sides of this equation we get:

$$a_i^{(n+1)} = \sum_{j=0}^{i} (a_j^{(n)} b_{i-j}^{(1)} + b_j^{(n)} a_{i-j}^{(1)}). \tag{34a}$$

Similarly we get:

$$b_i^{(n+1)} = \sum_{j=0}^{i} (b_j^{(n)} b_{i-j}^{(1)} - a_j^{(n)} a_{i-j}^{(1)}), \tag{34b}$$

$$c_i^{(n+1)} = \sum_{j=0}^{i} (c_j^{(n)} d_{i-j}^{(1)} + d_j^{(n)} c_{i-j}^{(1)}), \tag{34c}$$

$$d_i^{(n+1)} = \sum_{j=0}^{i} (d_j^{(n)} d_{i-j}^{(1)} + c_j^{(n)} c_{i-j}^{(1)}). \tag{34d}$$

Eqs. (34) are the required recurrent relations of the a's, b's , c's and d's coefficients for two consequent vales of n.

5.4. Symbolic expression of τ in term of t

In order to find the symbolic expression of the pseudo-time τ in terms of physical time t, we need an algorithm for reversing a power series and this will be developed in Subsection 5.4.1 while in Subsection 5.4.2 the algorithm will be applied to develop the required symbolic expression.

5.4.1. Series inversion of power series

General algorithm for reversing a power series will be developed as follows: Consider the functional equation:

$$\eta = \varsigma + \gamma\, \phi(\eta), \qquad |\gamma| < 1, \tag{35}$$

then according to Lagrange expansion theorem (Smart 1961), we have

$$\eta = \varsigma + \sum_{n=1}^{\infty} \frac{\gamma^n}{n!} \frac{d^{n-1}}{d\varsigma^{n-1}} [\phi(\varsigma)]^n. \tag{36}$$

Let $h(\theta)$ be a function which can be expressed in a Taylor series in the neighborhood of $\theta = \theta_0$. Thus:

$$h(\theta) = h_0 + \sum_{j=1}^{\infty} \frac{B_j}{j!} (\theta - \theta_0)^j, \tag{37}$$

where:

$$B_j = \frac{d^j h(\theta)}{d\theta^j}|_{\theta=\theta_0}.$$

In what follows, we assume that B_1 is different from zero and write Eq. (37) in the form:

$$\theta = \theta_0 + (h - h_0)\phi\ (\theta\ ), \tag{38}$$

where $\phi(\theta)$ is defined by:

$$\phi(\theta) = \frac{1}{B_1 + \sum_{j=1}^{\infty} [B_{j+1}/(j+1)!](\theta - \theta_0)^j}. \tag{39}$$

Eq. (38) is precisely the form of Eq. (35) and we can express θ as a power series in $\gamma = h - h_0$ to get:

$$\theta(h) = \theta_0 + \sum_{n=1}^{\infty} \frac{C_n}{n!} (h - h_0)^n, \tag{40}$$

where:

$$C_n = \frac{d^{n-1}}{d\theta^{n-1}} [\phi(\theta)]^n |_{\theta=\theta_0} \tag{41}$$

and $\phi(\theta)$ is defined in Eq. (39). The series for $\theta(h)$ is said to be the *reverse* of the series for $h(\theta)$.

Battin (1999) developed an elegant algorithm to express L coefficients of the coefficients $C_1, C_2, ...$ of the reversed series in terms of the coefficients $B_1, B_2, ...$of the original series. The basic equations of this algorithm are:

$$D_0^1|_{\theta=\theta_0} = \phi^{(0)}(\theta_0) \equiv \phi_0^{(0)} \equiv \frac{1}{B_1} \tag{42}$$

$$\frac{\mathrm{d}^k \phi(\theta)}{\mathrm{d}\theta^k} \equiv \phi_0^{(k)} = -\frac{1}{B_1} \sum_{i=1}^{k} \frac{1}{i+1} \begin{pmatrix} k \\ i \end{pmatrix} B_{i+1}\, \phi_0^{(k-i)}\ , \quad k = 1, 2, ..., L{-}1, \tag{43}$$

$$D_k^L = \frac{\mathrm{d}^k}{\mathrm{d}x^k} [\phi(\theta)]^L = L \sum_{j=0}^{k-1} \begin{pmatrix} k-1 \\ j \end{pmatrix} D_j^{L-1} \phi_0^{(k-j)}\ , \tag{44}$$

$$C_L = D_{L-1}^L|_{x=x_0}\ . \tag{45}$$

5.4.2. The inverse series of t

From Subsection (4.1) we see that the physical time t can be represented as power series in the pseudo-time τ as:

$$t = \sum_{i=0}^{\infty} t_i^{(n)} \tau^i\ .$$

Let this series be truncated at the term τ^L. To simplify the notations, here the upper subscripts (n) on the coefficients will be omitted, so:

$$t = \sum_{i=0}^{L} t_i \tau^i\ . \tag{46}$$

The inverse series of the physical time t is to find the pseudo-time τ in terms of t. Bellow we present an algorithm to illustrate the solution of this problem.

Algorithm

Purpose: to compute the C's coefficients of the power series representation of τ in terms of t, such that $\tau = \sum_{j=1}^{L} C_j t^j$.

Input: t_i ; $i = 1, 2, \cdots, L$.

Computational steps:

(1) Set $B_i = i!\ t_i\ \forall i = 0, 1, \cdots,\ L$.

(2) Apply the algorithm of Subsection 5.4.1 to find the C's coefficients of the series expansion $\sum_{k=1}^{L_t} \mathrm{C}_k t^k$.

(3) $C_j = C_j / j!$; $j = 1, 2, \cdots, L$.

(4) End.

As an example for $L = 5$ we get:

$$C_1 = 1/t_1,$$

$$C_2 = -t_2/t_1^3,$$

$$C_3 = (2t_2^2 - t_1 t_3)/t_1^5,$$

$$C_4 = (-5t_2^3 + 5t_1 t_2 t_3 - t_1^2 t_4)/t_1^7,$$

$$C_5 = (14\, t_2^4 - 21\, t_1 t_3 t_2^2 + 6t_1^2 t_2 t_4 + 3t_1^2 t_3^2 - t_1^3 t_5)/t_1^9.$$

Using the values of t_1, t_2 and t_3 as given in Table 1, we get for C_1, C_2 and C_3 the symbolic expressions listed in Table 3 for any values of n and ν.

Table 3: Symbolic expressions of C_1, C_2 and C_3 for any n and ν

$$C_1 = \frac{1}{\nu^2(d_0^2 - b_0^2)}\ ,$$

$$C_2 = \frac{2(a_0 b_0 \xi_0 + c_0 d_0 \eta_0)}{\nu^4 (b_0^2 - d_0^2)^3},$$

$$\begin{aligned} C_3 = 3 \big(&12 a_0 b_0 c_0 d_0 \eta_0 \xi_0 + a_0^2 b_0^2 d_0^2 J \nu^2 + 12 a_0 b_0^3 d_0^2 \eta_0 \nu^2 - \\ &- 6 a_0 b_0 d_0^4 \eta_0 \nu^2 + 6 a_0^2 b_0^5 d_0 \mu \nu^4 - 8 a_0^2 b_0^3 d_0^3 \mu \nu^4 + \\ &+ 2 a_0^2 b_0 d_0^5 \mu \nu^4 - 2 a_0^2 b_0 d_0^2 \mu \nu - 3 a_0^2 b_0^5 d_0 \nu^4 + 4 a_0^2 b_0^3 d_0^3 \nu^4 - \\ &- a_0^2 b_0 d_0^5 \nu^4 - 2 a_0^2 b_0^4 d_0^2 \nu + a_0^2 b_0 d_0^2 \nu - a_0^2 b_0^4 J \nu^2 - \\ &- 6 a_0 b_0^5 \eta_0 \nu^2 + 2 a_0^2 b_0^3 \mu \nu + 2 a_0^2 b_0^6 \nu - a_0^2 b_0^3 \nu + 5 a_0^2 b_0^2 \xi_0^2 + \\ &+ a_0^2 d_0^2 \xi_0^2 - b_0^2 c_0^2 d_0^2 J \nu^2 + 4 b_0^3 c_0^2 d_0^3 \mu \nu^4 - 4 b_0 c_0^2 d_0^5 \mu \nu^4 - \\ &- 2 b_0^3 c_0^2 d_0^3 \nu^4 + 2 b_0 c_0^2 d_0^5 \nu^4 + 6 b_0^4 c_0 d_0 \nu^2 \xi_0 - \\ &- 12 b_0^2 c_0 d_0^3 \nu^2 \xi_0 + 2 b_0^2 c_0^2 d_0^4 \nu + b_0^2 c_0^2 d_0 \nu + b_0^2 c_0^2 \eta_0^2 + \\ &+ b_0^2 d_0^2 \eta_0^2 - b_0^2 d_0^2 \xi_0^2 + b_0^4 \xi_0^2 + c_0^2 d_0^4 J \nu^2 + 5 c_0^2 d_0^2 \eta_0^2 + \\ &+ 6 c_0 d_0^5 \nu^2 \xi_0 - 2 c_0^2 d_0^6 \nu - c_0^2 d_0^3 \nu - d_0^4 \eta_0^2 \big) / \nu^6 (d_0^2 - b_0^2)^5. \end{aligned}$$

6. CONCLUSION

In concluding the present paper, we stress that:

- Explicit analytical solutions in power series of the pseudo–time τ for positions and velocities in regularized planes were obtained.
- Explicite analytical solutions in power series of the pseudo–time τ for positions and velocities in physical planes were obtained.
- Relations between coefficients of the power series were obtained for two consequent values of n.
- We developed explicite analytical solutions in power series form for the inverse problem to find τ in terms of t.
- Three symbolic expressions for coefficients of the power series were developed in terms of initial values.
- An efficient method depending on the continued fraction theory was provided.

The importance of this analytical power series representation is that it is invariant under many operations because the addition, multiplication, exponentiation, integration, differentiation, etc. of a power series is also a power series. This is the fact which provides excellent flexibility in dealing with analytical as well as computational developments of our problem.

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ГЛОБАЛНИ МЕТОД РЕГУЛАРИЗАЦИЈЕ ЗА ОГРАНИЧЕНИ ПРОБЛЕМ ТРИ ТЕЛА У РАВНИ

M. A. Sharaf[1] and H. R. Dwidar[2]

[1]*Department of Astronomy, Faculty of Science King Abdulaziz University, Jeddah, KSA*
E–mail: *sharaf_adel@hotmail.com*

[2]*Astronomy, Space and Meteorology Science Dept., Faculty of Science, Cairo University,Giza, Egypt*
E–mail: *hanyryd@cu.edu.eg*

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Оригинални научни рад

У овом раду предложен је метод опште регуларизације за ограничени проблем три тела у равни, заснован на трансформацији $z = x + iy = \nu \cos n(u + iv)$, где је $i = \sqrt{-1}$, $0 < \nu \leq 1$ и n природан број. Метод је развијен аналитички и рачунски. Код аналитичког извођења, добијена су тачна аналитичка решења у облику степеног реда по псеудо-времену τ за векторе положаје и брзине (u, v, u', v') и $(x, y, \dot{x}, \dot{y})$ у регуларизованој и у физичкој равни. Физичко време t је такође добијено као степени ред по τ. Осим тога, добијене су везе између коефицијената степеног реда за две узастопне вредности n. Такође, извели смо тачно аналитичко решење у облику степеног реда за инверзни проблем како бисмо τ изразили преко t. Као типични примери, изведена су три симблична израза за коефицијенте степеног реда изражена преко почетних вредности. Код рачунског метода, изведене су опште регуларизоване једначине кретања дате заједно са њиховим почетним вредностима у облицима погодним за нумеричко решавање коришћењем било ког програма за решавање диференцијалних једначина. Са друге стране, за нумерички развој степеног реда дат је ефикасан метод заснован на теорији верижног разломка.