Wavelet Cascade Applet: Mathematical Background

Cascading (or subdivision) is one of the standard methods to build wavelets and scaling functions. With this Applet you can graph all scaling functions and wavelets that satisfy refinement relations with four coefficients. The scaling function is defined on [0,3] and satisfies

$$\varphi(x) = c(0)\varphi(2x) + c(1)\varphi(2x-1) + c(2)\varphi(2x-2) + c(3)\varphi(2x-3)$$

and

$$\int_0^3 \varphi(x)\,dx = 1.$$

The associated (QMF) wavelet is given by

$$\psi(x) = c(3)\varphi(2x) - c(2)\varphi(2x-1) + c(1)\varphi(2x-2) - c(0)\varphi(2x-3)$$

It is well known that a continuous solution can only exist in case the refinement coefficients satisfy:

$$c(0) + c(2) = c(1) + c(3) = 1.$$

This leaves us with two degrees of freedom. We choose them to be the first coefficient even and the last one odd. We then have

$$c(0) = \mathit{even}, \quad c(1) = 1 - \mathit{odd}, \quad c(2) = 1 - \mathit{even}, \quad c(3) = \mathit{odd}.$$

Several properties of the scaling function immediately follow from the refinement coefficients:

- In case $\mathit{even} = \mathit{odd}$, the scaling function is symmetric and the wavelet is anti-symmetric.
- By flipping *even* and *odd*, the function flips around $x = 3/2$.
- In case $\mathit{even}(1-\mathit{even}) + \mathit{odd}(1-\mathit{odd}) = 0$, the scaling function and wavelet are orthogonal. In the $(\mathit{even}, \mathit{odd})$ plane this is a circle which goes trough the corners of the unit square.
- In case $\mathit{even} + \mathit{odd} = 1/2$, the order of the scaling functions is two, i.e., the scaling function and its translates can reproduce linears. In the orthogonal case, the wavelet then has two vanishing moments.
- In case *even* or *odd* is zero, then the support of the function is $[0,2]$ or $[1,3]$ respectively. The function then also is interpolating in the sense that it takes the value 1 at $x = 1$ or $x = 2$ respectively and zero at the other integers.
- In case $0 < \mathit{even} < 1$ and $0 < \mathit{odd} < 1$, i.e., the red dot is in the unit square, the cascade algorithm will only use convex combinations. As a result the scaling function will only take on values between 0 and 1, while the wavelet only takes values between -1 and 1.

- A lot of research has been done on how the smoothness of the scaling function depends on the refinements coefficients. In fact the cascade algorithm is an instance of a 1D subdivision algorithm. For a fixed length, the smoothest solution is always the B-spline. Colella and Heil study the four coefficient case in detail and are able to outline the area in the (even,odd) plane where the scaling function is continuous. This area is draw shaded in light gray. For more information, check D. Colella and C. Heil, Characterizations of scaling functions: Continuous Solutions, SIAM J. Matrix Anal. Appl., vol. 15, pp. 496-518, 1994. The idea for this applet is partly inspired by their work. Special thanks to Chris Heil for providing the polygon outlining the area of continuity.

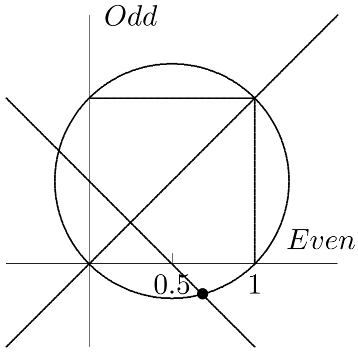
Several well-known functions are part of this class.

- If *even* = 1 and *odd* = 0, then the scaling function is a box function which is one on the interval $[0, 1]$ and zero elsewhere. Box functions on $[1, 2]$ and $[2, 3]$ have respectively *even* = *odd* = 0 and *even* = 0 and *odd* = 1. The box functions are orthogonal and lie on 3 of the corners of the unit square. These are the only scaling functions that are orthogonal and symmetric. A box scaling function leads to a Haar wavelet.
- If *even* = 1/2 and *odd* = 0 the refinement coeffients are $(\,1/2 \quad 1 \quad 1/2 \quad 0\,)$, and the scaling function is a hat function (linear B-spline).
- If *even* = *odd* = 1/4 the refinement coefficients are $(\,1/4 \quad 3/4 \quad 3/4 \quad 1/4\,)$ and the scaling function is a quadratic B-spline. The quadratic B-spline has order 3.
- If $even = (1+\sqrt{3})/4$ and $odd = 1/2 - even$, you get the Daubechies D4 orthogonal scaling function and wavelet. This and its flipped version are the only solutions which are orthogonal and have order 2.
- If *even* = *odd* = 1, the cascade algorithm diverges. However, in a weak (distributional) sense, it converges and the scaling function is 1/3 times the indicator function on $[0, 3]$.

Function

⊙ Scaling

○ Wavelet



none ⊙
symetric ○
orthogonal ○
order 2 ○
interpolating ○

Haar | linear | spline | Daubechies | zebra