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LEXICOGRAPHIC POLYNOMIALS OF GRAPHS AND THEIR SPECTRA

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In honor of Dragoš Cvetković on the occasion of his 75th birthday.

For a (simple) graph H and non-negative integers $c_0, c_1, \dots, c_d$ ($c_d \neq 0$), $p(H) = \sum_{k=0}^{d} c_k \cdot H^k$ is the lexicographic polynomial in H of degree d, where the sum of two graphs is their join and $c_k \cdot H^k$ is the join of c_k copies of H^k. The graph H^k is the kth power of H with respect to the lexicographic product ($H^0 = K_1$). The spectrum (if H is connected and regular) and the Laplacian spectrum (in general case) of $p(H)$ are determined in terms of the spectrum of H and c_k's. Constructions of infinite families of cospectral or integral graphs are announced.

1. INTRODUCTION

Let $G = (V_G, E_G)$ be a simple graph (so, without loops or multiple edges), V_G its vertex set, and E_G its edge set. The cardinality of V_G is called the *order* of G. If $|V_G| = n$ then, if not told otherwise, we assume that $V_G = \{1, 2, \dots, n\}$. If $e \in E_G$, we write $e = ij$, where i and j are its end-vertices. If $i \in V_G$, then $N_G(i) = \{j : j \sim i\}$ is the set of neighbors of i, while $\deg_G(i) = |N_G(i)|$ is its vertex degree. The graph is called *regular* if all its vertices have the same degree. If this degree is r, then we say that the graph is r*-regular*.

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The *adjacency matrix* of G, denoted by A_G, is the square matrix of the same order as G, with (i, j)th entry equal to 1 if $ij \in E_G$, and 0 otherwise. The eigenvalues and the spectrum of A_G are called the *eigenvalues* and the *spectrum* of G. Since A_G is real and symmetric, its eigenvalues are real. The eigenvalues of G (in non-increasing order) are denoted by $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$. If G is connected, the largest eigenvalue $\lambda_1(G)$ is simple, while if G is r-regular, it is equal to r.

The *Laplacian matrix* of G is the matrix $L_G = D_G - A_G$, where D_G is the diagonal matrix of (vertex) degrees. The eigenvalues and the spectrum of L_G are called the *Laplacian eigenvalues* and the *Laplacian spectrum* (for short, *L-eigenvalues* and *L-spectrum*) of G. The Laplacian matrix is symmetric but also positive semidefinite, and so its eigenvalues are real and non-negative. We denote the L-eigenvalues of G (in non-increasing order) by $\mu_1(G), \mu_2(G), \ldots, \mu_n(G)$. Recall, $\mu_n(G) = 0$ for any G, while $\mu_{n-1}(G)$ is called (by Fiedler) the *algebraic connectivity* of G [**7**].

In general, for any symmetric matrix M, we denote its spectrum (which is a multiset) by

$$\sigma(M) = \left\{ \gamma_1^{[g_1]}, \gamma_2^{[g_2]}, \ldots, \gamma_s^{[g_s]} \right\}.$$

where $\gamma_1 > \gamma_2 > \cdots > \gamma_s$ are all distinct eigenvalues of M, while the entries in the upper brackets stand for the multiplicities. In particular, we write $\sigma_A(G)$ for $\sigma(A_G)$ and $\sigma_L(G)$ for $\sigma(L_G)$. For more details on the spectral graph theory see [**2**] or [**4, 5**]; in addition, see the nice survey articles [**12, 13**] on early results for Laplacian spectrum and its applications.

The *complement* of a graph $G = (V_G, E_G)$ is the graph $\bar{G} = (V_{\bar{G}}, E_{\bar{G}})$ with $V_{\bar{G}} = V_G$ and $ij \in E_{\bar{G}}$ if and only if $ij \notin E_G$. The *union* of two disjoint graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$ is the graph $G \cup H = (V_G \cup V_H, E_G \cup E_H)$. The *join* of the above (disjoint) graphs is the graph $G + H = (V_G \cup V_H, E_G \cup E_H \cup \{ij : i \in V_G, j \in V_H\}$. Clearly, $\overline{G + H} = \bar{G} \cup \bar{H}$. The union or the join of more than two mutually disjoint graphs can be defined in a similar way. For a non-negative integer c, we denote by $c\,G$ the union of c disjoint copies of G and by $c \cdot G$ the join of c disjoint copies of G (the union and the join of 0 copies of G, $0G$ and $0 \cdot G$, respectively, are both the empty graph K_0, i.e., the graph with an empty set of vertices).

The *lexicographic product* of graphs was introduced in 1959 by Harary [**10**], and independently, in the same year, by Sabidussi [**15**]. For two graphs G and H, their lexicographic product $G[H]$ is the graph whose vertex set is the Cartesian product $V_G \times V_H$, with two vertices (i_1, j_1) and (i_2, j_2) being adjacent whenever $i_1 i_2 \in E_G$, or $i_1 = i_2$ and $j_1 j_2 \in E_H$. It is easy to see that this graph operation is associative, but not commutative (see [**9, 11**] for more details).

The lexicographic product is generalized in the following way [**16**]. Let G be a graph of order n, and let $H_1, H_2, \ldots, H_n$ be an arbitrary collection of n disjoint graphs. The *generalized lexicographic product* $G[H_1, H_2, \ldots, H_n]$ (or *G-join* of graphs $H_1, H_2, \ldots, H_n$, according to [**3**]) is the graph whose vertex and edge sets are

$$V_{G[H_1,H_2,\dots,H_n]} = \bigcup_{i=1}^{n} V_{H_i} \quad \text{and} \quad E_{G[H_1,H_2,\dots,H_n]} = \bigcup_{i=1}^{n} E_{H_i} \bigcup \{\cup_{ij\in E_G} E_{H_i+H_j}\}.$$

The spectrum of $G[H_1, H_2, \dots, H_n]$ is computed in [**3, 16**], in terms of the spectra of the regular graphs $H_1, H_2, \dots, H_n$ and the spectrum of a (weighted) adjacency matrix associated to graph G. In [**3, 8**], using different approaches, the L-spectrum of $G[H_1, H_2, \dots, H_n]$ is computed in terms of the Laplacian spectrum of arbitrary graphs $H_1, H_2, \dots, H_n$ and the spectrum of a (weighted) Laplacian matrix associated to graph G. Besides, it is worth mentioning that in [**14**] the Laplacian spectrum of the generalized lexicographic product $T[H_1, H_2, \dots, H_n]$, where T is a tree, is determined.

As we announced in the Abstract, in this paper we derive the spectrum (of the adjacency matrix) of the lexicographic polynomial $p(H)$ of a regular graph H as well as the Laplacian spectrum of $p(H)$ of an arbitrary graph H. Furthermore, we indicate certain constructions of cospectral graphs and integral graphs.

The next section is the preparatory one. The main results are given in two subsections of Section 3, one covering the (adjacency) spectrum, the other the L-spectrum of graphs. Concluding remarks are given in the last section.

2. SOME PRELIMINARY RESULTS

We start with a particular case of [**3**, Theorem 5].

Lemma 1. *Let $G_1, G_2, \dots, G_d$ be regular graphs of orders $n_1, n_2, \dots, n_d$, respectively. Then*

$$\sigma_A(K_d[G_1, G_2, \dots, G_d]) = \left(\bigcup_{k=1}^{d} \sigma_A(G_k) \setminus \{\lambda_1(G_k)\}\right) \cup \sigma(C),$$

where

$$C = \begin{pmatrix} \lambda_1(G_1) & \sqrt{n_1 n_2} & \cdots & \sqrt{n_1 n_d} \\ \sqrt{n_2 n_1} & \lambda_1(G_2) & \cdots & \sqrt{n_2 n_d} \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{n_d n_1} & \sqrt{n_d n_2} & \cdots & \lambda_1(G_d) \end{pmatrix}.$$

The following lemma is another particular case of the same result (see also [**1**, Corollary 2.2]).

Lemma 2. *If G is a graph of order n_1 with the spectrum*

$$\sigma_A(G) = \left\{\gamma_1^{[g_1]}(G), \gamma_2^{[g_2]}(G), \dots, \gamma_s^{[g_s]}(G)\right\}$$

and if H is a connected r-regular graph of order n_2 with the spectrum

$$\sigma_A(H) = \left\{r, \gamma_2^{[h_2]}(H), \ldots, \gamma_t^{[h_t]}(H)\right\},$$

then

$$\sigma_A(G[H]) = \{(n_2\gamma_1(G)+r)^{[g_1]}, \ldots, (n_2\gamma_s(G)+r)^{[g_s]}\} \cup \{\gamma_2^{[n_1h_2]}(H), \ldots, \gamma_t^{[n_1h_t]}(H)\}.$$

In what follows, let H^k be the kth power of H with respect to the lexicographic product (with $H^0 = K_1$). We first recall the following result (see [**1**, Corollary 3.4]).

Theorem 1. *Let H be a connected r-regular graph of order n with the spectrum*

$$\sigma_A(H) = \{r, \gamma_2^{[h_2]}(H), \ldots, \gamma_t^{[h_t]}(H)\},$$

Then, for each integer $k \geq 1$, H^k is an r_k-regular graph of order n^k, such that

$$\begin{aligned} r_k &= r\frac{n^k-1}{n-1} \quad \textit{and} \\ \sigma_A(H^k) &= \left(\bigcup_{i=0}^{k-1} \{(n^i\gamma_2(H)+r_i)^{[n^{k-1-i}h_2]}, \ldots, (n^i\gamma_t(H)+r_i)^{[n^{k-1-i}h_t]}\}\right) \cup \{r_k\}. \end{aligned}$$

Remark 1. *If H is a graph as specified in the previous theorem, then*

$$\begin{aligned} \sigma_A(H^{k+1}) &= \left(\bigcup_{i=0}^{k} \{(n^i\gamma_2(H)+r_i)^{[n^{k-i}h_2]}, \ldots, (n^i\gamma_t(H)+r_i)^{[n^{k-i}h_t]}\}\right) \cup \{r_{k+1}\}, \\ &= \left(\bigcup_{i=0}^{k-1} \{(n^i\gamma_2(H)+r_i)^{[n^{k-i}h_2]}, \ldots, (n^i\gamma_t(H)+r_i)^{[n^{k-i}h_t]}\}\right) \bigcup \\ &\quad \{(n^k\gamma_2(H)+r_k)^{[h_2]}, \ldots, (n^k\gamma_t(H)+r_k)^{[h_t]}\} \bigcup \{r_{k+1}\} \\ &= \left(\bigcup_{i=0}^{k-1} \{(n^i\gamma_2(H)+r_i)^{[n^{k-1-i}h_2]}, \ldots, (n^i\gamma_t(H)+r_i)^{[n^{k-1-i}h_t]}\}\right) \bigcup \\ &\quad \left(\bigcup_{i=0}^{k-1} \{(n^i\gamma_2(H)+r_i)^{[(n-1)n^{k-1-i}h_2]}, \ldots, (n^i\gamma_t(H)+r_i)^{[(n-1)n^{k-1-i}h_t]}\}\right) \\ &\quad \bigcup \{(n^k\gamma_2(H)+r_k)^{[h_2]}, \ldots, (n^k\gamma_t(H)+r_k)^{[h_t]}\} \bigcup \{r_{k+1}\} \\ &= \left(\sigma_A(H^k) \setminus \{r_k\}\right)^{[n]} \bigcup \{(n^k\gamma_2(H)+r_k)^{[h_2]}, \ldots, (n^k\gamma_t(H)+r_k)^{[h_t]}\} \\ &\quad \bigcup \{r_{k+1}\}, \end{aligned}$$

where $\big(\sigma_A(H^k)\setminus\{r_k\}\big)^{[n]}$ *means that this spectrum is repeated* n *times. Therefore,* r_{k+1} *and* $n^k\gamma_2(H)+r_k,\dots,n^k\gamma_t(H)+r_k$ *are the only eigenvalues in* $\sigma_A(H^{k+1})$ *which can be distinct from the ones in* $\sigma_A(H^k)$*.*

Finally, if ξ *stands for the number of distinct eigenvalues of a graph, then* $\xi(H^{k+1})\le\xi(H^k)-1+t$*.*

Now, a simple consequence reads as follows.

Corollary 1. *Let* H *be a connected* r*-regular graph of order* n *with the spectrum*

$$\sigma_A(H)=\{r,\gamma_2^{[h_2]}(H),\dots,\gamma_t^{[h_t]}(H)\}.$$

Then, for every $k\ge 1$*, we have*

$$\begin{aligned}\sigma_A(H^{k+1}) &= \big(\sigma_A(H^k)\setminus\{r_k\}\big)^{[n]}\bigcup\{(n^k\gamma_2(H)+r_k)^{[h_2]},\dots,(n^k\gamma_t(H)+r_k)^{[h_t]}\}\\ &\quad \bigcup\{r_{k+1}\}.\end{aligned}$$

and

$$\xi(H^{k+1})\le\xi(H)+k(t-1).$$

Proof Both parts follow from Remark 1. The first follows directly, while the second is obtained by the next chain of inequalities

$$\xi(H^{k+1})\le\xi(H^k)-1+t\le\xi(H^{k-1})-2+2t\le\cdots\le\xi(H)+k(t-1).$$

□

In what follows we shall also need the spectrum of $G[H^k]$, when G is a complete graph. This is obtained by applying Lemma 2 and Theorem 1 again.

Corollary 2. *Let* H *be a connected* r*-regular graph of order* n *with the spectrum*

$$\sigma_A(H)=\{r,\gamma_2^{[h_2]}(H),\dots,\gamma_t^{[h_t]}(H)\}.$$

If k *and* p *are positive integers, then*

$$\begin{aligned}\sigma_A(K_p[H^k]) &= \{(n^k(p-1)+r_k)^{[1]},(r_k-n^k)^{[p-1]}\}\\ &\quad \bigcup\left(\bigcup_{i=0}^{k-1}\{(n^i\gamma_2(H)+r_i)^{[pn^{k-1-i}h_2]},\dots,(n^i\gamma_t(H)+r_i)^{[pn^{k-1-i}h_t]}\}\right),\end{aligned}$$

where $r_k=r\frac{n^k-1}{n-1}$*.*

Finally, concerning the L-spectrum we have the following lemma (see [**1**, Corollary 3.10]).

Lemma 3. *Let H be a graph of order n with the L-spectrum*

$$\sigma_L(H) = \{\mu_1(H), \mu_2(H), \ldots, \mu_n(H)\}.$$

Then H^k is a graph of order n^k and, for every $k \geq 2$, we have

$$\begin{aligned}\sigma_L(H^k) \quad &= \quad \bigcup_{i=1}^{k-1} \left(\bigcup_{(j_i,\ldots,j_{k-1}) \in V_H^{k-i}} \left\{ n^{i-1}\mu_l(H) + \sum_{s=i}^{k-1} n^s \deg_H(j_s) : 1 \leq l \leq n-1 \right\} \right) \\ &\quad\quad \bigcup \left\{ n^{k-1}\mu_j(H) : 1 \leq j \leq n \right\}.\end{aligned}$$

3. LEXICOGRAPHIC POLYNOMIALS OF GRAPHS

For any graph H and non-negative integers $c_0, c_1, \ldots, c_d$ ($c_d \neq 0$), we can consider the graph $p(H)$ of the form

$$(1) \qquad \sum_{k=0}^{d} c_k \cdot H^k = c_0 \cdot H^0 + c_1 \cdot H^1 + \cdots + c_d \cdot H^d,$$

where H^k is a lexicographic power of H (in the sequel, we assume that $H^0 = K_1$ where K_1 denotes the graph of order 1). The expression (1) is called the *lexicographic polynomial* in H of degree d. Throughout the paper we assume that $d > 0$ and $p(H)$ is not a monomial with $c_d = 1$ (that is, $p(H) \neq H^d$).

It is immediate that two lexicographic polynomials $p_1(H)$ and $p_2(H)$ can be summed (by the join), multiplied by a scalar, and consequently multiplied mutually. Precisely, assuming that $p(H) = \sum_{k_1=0}^{d_1} a_{k_1} \cdot H^{k_1}$ and $q(H) = \sum_{k_2=0}^{d_2} b_{k_2} \cdot H^{k_2}$, we have

$$p(H) \cdot q(H) = \sum_{0 \leq k_1 \leq d_1,\ 0 \leq k_2 \leq d_2} (a_{k_1} b_{k_2}) \cdot H^{k_1+k_2}.$$

Remark 2. *It is worth mentioning that even when the graph H is regular, its lexicographic polynomial* (1)*, in general, need not be regular. For example, the lexicographic polynomial $p(H) = H^0 + H^1$, where $H = \bar{K}_n$, is the star $K_{1,n}$.*

Recall that the *independence number* $\alpha(G)$ (the *clique number* $\omega(G)$) of a graph G is the maximal order of a co-clique (resp. clique) contained in G as an induced subgraph (note that a co-clique and clique of the same order are complementary pairs of graphs). Let $\mathrm{diam}(G)$ be the diameter of G. The next proposition provides a few properties of lexicographic polynomials.

Theorem 2. *Let H be a graph of order n and $p(H)$ a lexicographic polynomial in H of degree d. Then*

(i) $\sum_{k=0}^{d} c_k n^k$ *is the order of* $p(H)$*;*

(ii) $c_k \cdot H^k = K_{c_k}[H^k]$, *for every* $k = 0, 1, \ldots, d$;

(iii) $p(H) = \sum_{k=0}^{d} K_{c_k}[H^k] = K_{d+1}[K_{c_0}[H^0], K_{c_1}[H^1], \ldots, K_{c_d}[H^d]]$;

(iv) *If* H *is connected but non-complete, then* $\operatorname{diam}(p(H)) = 2$;

(v) $\alpha(p(H)) = \alpha(H)^d$ *and* $\omega(p(H)) = \omega(H)^d$.

Proof We have:

(i) Since the order of the join of two graphs is the sum of their orders, the result follows by the fact that the order of $c_k \cdot H^k$ is $c_k n^k$, for $k = 0, 1, \ldots, d$.

(ii) Observe that $c_k \cdot H^k$ is the join of c_k copies of H^k, and thus it is equal to $K_{c_k}[H^k]$.

(iii) Since $p(H)$ is obtained as the join of graphs from the set $\{c_0 \cdot H^0, c_1 \cdot H^1, \ldots, c_d \cdot H^d\}$, the result follows from (ii).

(iv) The property follows immediately since we have that $p(H) = K_{d+1}[K_{c_0}[H^0], K_{c_1}[H^1], \ldots, K_{c_d}[H^d]]$.

(v) This property follows from (iv), taking into account that $\alpha(H^k) = (\alpha(H))^k$ and $\omega(H^k) = (\omega(H))^k$ (see [**1**]). □

4. SPECTRUM OF A LEXICOGRAPHIC POLYNOMIAL OF A REGULAR GRAPH

Using Lemma 1 and Corollary 2, we can determine the spectrum of the lexicographic polynomial $p(H)$ whenever the graph under consideration is regular.

Theorem 3. *Let* H *be a connected* r*-regular graph of order* n *with the spectrum*

$$\sigma_A(H) = \{r, \gamma_2^{[h_2]}, \ldots, \gamma_t^{[h_t]}\}.$$

Then

$$\sigma_A(p(H)) = \left(\bigcup_{k=0}^{d} \sigma_A(K_{c_k}[H^k]) \setminus \{r_k + (c_k - 1)n^k\} \right) \bigcup \sigma(C^*), \tag{2}$$

where, for $k = 0, 1, \ldots, d$, $r_k = r\frac{n^k - 1}{n-1}$,

$$\sigma_A(K_{c_k}[H^k]) \setminus \{r_k + (c_k - 1)n^k\} = \tag{3}$$

$$\bigcup_{i=0}^{k-1} \{(n^i\gamma_2(H) + r_i)^{[c_k n^{k-1-i} h_2]}, \ldots, (n^i\gamma_t(H) + r_i)^{[c_k n^{k-1-i} h_t]}\} \bigcup \{(r_k - n^k)^{[c_k - 1]}\},$$

and C^* is the matrix obtained from

$$(4)\qquad C=\begin{pmatrix} r_0+(c_0-1)n^0 & \sqrt{c_0c_1n^{0+1}} & \cdots & \sqrt{c_0c_dn^{0+d}}\\ \sqrt{c_1c_0n^{1+0}} & r_1+(c_1-1)n & \cdots & \sqrt{c_1c_dn^{1+d}}\\ \vdots & \vdots & \ddots & \vdots\\ \sqrt{c_dc_0n^{d+0}} & \sqrt{c_dc_1n^{d+1}} & \cdots & r_d+(c_d-1)n^d \end{pmatrix},$$

after deleting the columns and rows intersecting at a negative main diagonal entry.

Proof If there is a negative diagonal entry in C, say $r_i+(c_i-1)n^i$, then $c_i=0$ holds, so $c_i\cdot H^i=K_{c_i}[H^i]$ is the empty graph. Now, (2) is obtained from Lemma 1 by taking into account that

$$p(H)=\sum_{k=0}^{d}K_{c_k}[H^k]=K_{d+1}[K_{c_0}[H^0],K_{c_1}[H^1],\dots,K_{c_d}[H^d]],$$

while (3) follows from Corollary 2. □

Observe that matrix C does not depend on any structural parameter of H, apart from n and r. In other words, C is an invariant of a lexicographic polynomial applied to any r-regular graphs of fixed order. Consequently, if H_1 and H_2 are such regular graphs, then $p(H_1)$ and $p(H_2)$ share the part of the spectrum emerging from C.

We proceed with the following general result.

Lemma 4. *Let M be a square matrix of order $d+1$ $(d\geq 1)$ whose diagonal entries lie in the interval $(0,1)$ and all off-diagonal entries are 1's. Then M has one positive and d negative eigenvalues.*

Proof We have $M=A_{K_{d+1}}+D_\varepsilon$, where D_ε is the diagonal matrix consisting of diagonal entries of M, and ε stands for the maximum of those elements. Using the Courant-Weyl inequalities [**4, 5**], we obtain $\gamma_2(M)\leq\gamma_2(K_{n+1})+\gamma_1(D_\varepsilon)=-1+\varepsilon<0$. □

Therefore, we can deduce the following corollary.

Corollary 3. *The matrix C^* (defined in Theorem 3) of order d^*+1 has one positive and d^* negative eigenvalues.*

Proof Without loss of generality, we may assume that $C=C^*$, so that we have $c_k>0$ for all $k=0,1,\dots,d$. By setting $D'=\operatorname{diag}\left(\frac{1}{\sqrt{c_0n^0}},\frac{1}{\sqrt{c_1n^1}},\dots,\frac{1}{\sqrt{c_dn^d}}\right)$ and $M=D'CD'$, we obtain

$$M=\begin{pmatrix} m_0 & 1 & 1 & \cdots & 1\\ 1 & m_1 & 1 & \cdots & 1\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & 1 & \cdots & m_d \end{pmatrix},$$

where $m_k = \frac{r_k + (c_k - 1)n^k}{c_k n^k}$.

The matrices C and M are congruent. By the Sylvester law of inertia [**17**, Theorem 8.3], they have an equal number of positive and an equal number of negative eigenvalues. Therefore, it is sufficient to show that the claim holds for the matrix M. Since all c_k's are positive integers, we have that all m_k's are positive, and so, to apply the previous lemma, we need to show that all m_k's are less than 1. Since every m_k can be written as $m_k = 1 + \frac{r_k}{c_k n^k} - \frac{1}{c_k}$, we deduce that $m_k < 1$ is equivalent to $\frac{r_k}{n^k} < 1$, and the latter inequality follows from the fact that n^k is the order of the r_k-regular graph H^k (if necessary, see Theorem 1). □

Example 1. Let H be the Petersen graph and consider the lexicographic polynomial

$$p(H) = 5 \cdot H + 2 \cdot H^2 + 3 \cdot H^3$$

of order 3250. In this case we have $n = 10$ and $r = 3$. Since the spectrum of H is given by $\sigma_A(H) = \{3, 1^{[5]}, -2^{[4]}\}$, by applying Theorem 3 we obtain

$$\begin{aligned}\sigma_A(p(H)) &= \sigma_A(K_5[H]) \setminus \{3 + 4 \times 10\} \cup \sigma_A(K_2[H^2]) \setminus \{33 + (2-1)10^2\} \cup \\ &\quad \sigma_A(K_3[H^3]) \setminus \{333 + (3-1)10^3\} \cup \sigma(C^*),\end{aligned}$$

with

$$C^* = \begin{pmatrix} 43 & 100 & 100\sqrt{15} \\ 100 & 133 & 200\sqrt{15} \\ 100\sqrt{15} & 200\sqrt{15} & 2333 \end{pmatrix}.$$

So, we have

$$\begin{aligned}\sigma_A(p(H)) &= \{43, 1^{[25]}, -2^{[20]}, -7^{[4]}\} \setminus \{43\} \cup \{133, 13^{[10]}, 1^{[100]}, -2^{[80]}, -17^{[8]}, -67\} \setminus \{133\} \\ &\quad \cup \{2333, 133^{[15]}, 13^{[150]}, 1^{[1500]}, -2^{[1200]}, -17^{[120]}, -167^{[12]}, -667^{[2]}\} \\ &\quad \setminus \{2333\} \cup \sigma(C^*) \\ &= \{133^{[15]}, 13^{[160]}, 1^{[1625]}, -2^{[1300]}, -7^{[4]}, -17^{[128]}, -67, -167^{[12]}, -667^{[2]}\} \\ &\quad \cup \left\{1313 + 60\sqrt{489}, -117, 1313 - 60\sqrt{489}\right\}.\end{aligned}$$

We can locate the largest eigenvalue of a lexicographic polynomial of a regular graph as follows.

Theorem 4. *Let H be a connected r-regular graph of order n with the spectrum*

$$\sigma_A(H) = \left\{r, \gamma_2^{[h_2]}(H), \ldots, \gamma_t^{[h_t]}(H)\right\}.$$

Then the largest eigenvalue of the lexicographic polynomial $p(H) = \sum_{k=0}^{d} c_k \cdot H^k$ of an arbitrary graph H of order n is the largest eigenvalue of the matrix C^* defined in Theorem 3.

Proof It immediately follows from Theorem 3 that the largest eigenvalue of $p(H)$ is equal to $\max\{r_{d-1} + n^{d-1}\gamma_2(H), \rho\}$, where $r_{d-1} = r\frac{n^{d-1}-1}{n-1}$, while ρ stands for the largest eigenvalue of the matrix C^*.

From Corollary 3, we have $\rho > \mathrm{tr}(C^*) = \sum_{k=0}^{d}(\max\{0, r_k + (c_k - 1)n^k\})$. Therefore, it is sufficient to prove that $\sum_{k=0}^{d}(\max\{0, r_k + (c_k - 1)n^k\}) \geq r_{d-1} + n^{d-1}\gamma_2(H)$ holds. Indeed, $c_d > 0$ implies that

$$\begin{aligned}\sum_{k=0}^{d}(\max\{0, r_k + (c_k - 1)n^k\}) &= \sum_{k=0}^{d-1}(\max\{0, r_k + (c_k - 1)n^k\}) + r_d + (c_d - 1)n^d \\ &\geq r_d \\ &> r_{d-1} + n^{d-1}\gamma_2(H),\end{aligned}$$

where the last inequality is true since $r_d = r\frac{n^d-1}{n-1} = r\sum_{k=0}^{d-2} n^k + rn^{d-1} > r_{d-1} + n^{d-1}\gamma_2(H)$. □

5. L-SPECTRUM OF A LEXICOGRAPHIC POLYNOMIAL OF A GRAPH

For a graph G of order n its Laplacian eigenvalues are related to the Laplacian eigenvalues of its complement $\bar{G}$ in the following way. If the spectrum of G is $\sigma_L(G) = \{\mu_1(G), \mu_2(G), \ldots, \mu_n(G)\}$, then

$$\mu_k(\bar{G}) = n - \mu_{n-k}(G), \text{ for } 1 \leq k \leq n-1, \tag{5}$$

and $\mu_n(\bar{G}) = 0$.

Consider the lexicographic polynomial $p(H) = \sum_{k=0}^{d} c_k \cdot H^k$ of an arbitrary graph H of order n. Since $p(H) = K_{d+1}[K_{c_0}[H^0], K_{c_1}[H^1], \ldots, K_{c_d}[H^d]]$, it follows that its complement is given by

$$\begin{aligned}\overline{p(H)} &= \overline{K_{d+1}[K_{c_0}[H^0], K_{c_1}[H^1], \ldots, K_{c_d}[H^d]]} \\ &= \bigcup_{k=0}^{d} \overline{K_{c_k}[H^k]} = \bigcup_{k=0}^{d} c_k\overline{H^k} = \bigcup_{k=0}^{d} c_k\bar{H}^k,\end{aligned}$$

since $\overline{H^k} = \bar{H}^k$ (see [**1**]). Therefore, we have

$$\sigma_L(\overline{p(H)}) = \bigcup_{k=0}^{d} c_k\sigma_L(\overline{H}^k), \tag{6}$$

where $c\,\sigma$ denotes the multiset obtained by the union of c copies of the multiset σ.

We next have the following lemma.

Lemma 5. *Let H be a graph of order n with the L-spectrum*

$$\sigma_L(H) = \{\mu_1(H), \mu_2(H), \ldots, \mu_n(H)\}.$$

Then $c_0\sigma_L(\bar{H}^0) = \{0^{[c_0]}\}$, $c_1\sigma_L(\bar{H}^1) = \{\mu_i(\bar{H})^{[c_1]} : 1 \leq i \leq n\}$, and, for every integer $k \geq 2$,

$$\begin{aligned} c_k\sigma_L(\bar{H}^k) &= \bigcup_{i=1}^{k-1}\Bigg(\bigcup_{(j_i,\ldots,j_{k-1})\in V_H^{k-i}}\Bigg\{\Big(n^{i-1}(n-\mu_l(H)) + \sum_{s=i}^{k-1} n^s(n-1-\deg_H(j_s))\Big)^{[c_k]} \\ &\quad : 1 \leq l \leq n-1\Bigg\}\Bigg) \bigcup \Big\{\Big(n^{k-1}(n-\mu_j(H))\Big)^{[c_k]} : 1 \leq j \leq n-1\Big\} \cup \{0^{[c_k]}\}. \end{aligned}$$

Proof The expressions for $c_0\sigma_L(\bar{H}^0)$ and $c_1\sigma_L(\bar{H}^1)$ are immediate. For an integer $k \geq 2$, using Lemma 3 we obtain

$$\begin{aligned} \sigma_L(\bar{H}^k) &= \bigcup_{i=1}^{k-1}\Bigg(\bigcup_{(j_i,\ldots,j_{k-1})\in V_H^{k-i}}\Bigg\{n^{i-1}\mu_l(\bar{H}) + \sum_{s=i}^{k-1} n^s \deg_{\bar{H}}(j_s) : 1 \leq l \leq n-1\Bigg\}\Bigg) \\ &\quad \cup \big\{n^{k-1}\mu_j(\bar{H}) : 1 \leq j \leq n\big\} \\ &= \bigcup_{i=1}^{k-1}\Bigg(\bigcup_{(j_i,\ldots,j_{k-1})\in V_H^{k-i}}\Bigg\{n^{i-1}(n-\mu_{n-l}(H)) + \sum_{s=i}^{k-1} n^s(n-1-\deg_H(j_s)) \\ &\quad : 1 \leq l \leq n-1\Bigg\}\Bigg) \cup \big\{n^{k-1}(n-\mu_{n-j}(H)) : 1 \leq j \leq n-1\big\} \cup \{0\}. \end{aligned}$$

Finally, taking into account that the Laplacian eigenvalues of $c_k\bar{H}^k$ are the Laplacian eigenvalues of $\bar{H}^k$ repeated c_k times, the result follows. □

Applying this lemma we obtain the L-spectrum of a lexicographical polynomial.

Theorem 5. *Let H be a graph of order n with the L-spectrum*

$$\sigma_L(H) = \{\mu_1(H), \mu_2(H), \ldots, \mu_n(H)\}.$$

Then

$$\sigma_L(p(H)) = \left(\bigcup_{k=0}^{d} c_k(\nu - \sigma_L(\bar{H}^k))\right) \setminus \{\nu\} \cup \{0\},$$

where for $k = 0, 1, \ldots, d$, the multiset $c_k(\nu - \sigma_L(\bar{H}^k))$ is equal to

$$\begin{aligned} &\bigcup_{i=1}^{k-1}\Bigg(\bigcup_{(j_i,\ldots,j_{k-1})\in V_H^{k-i}}\Bigg\{\Big(\nu - \Big(n^{i-1}(n-\mu_l(H)) + \sum_{s=i}^{k-1} n^s(n-1-\deg_H(j_s))\Big)\Big)^{[c_k]} \\ &: 1 \leq l \leq n-1\Bigg\}\Bigg) \bigcup \Bigg\{\Big(\nu - \Big(n^{k-1}(n-\mu_j(H))\Big)\Big)^{[c_k]} : 1 \leq j \leq n-1\Bigg\} \cup \{\nu^{[c_k]}\}, \end{aligned}$$

while $\nu = \sum_{k=0}^{d} c_k n^k$ *is the order of* $p(H)$.

Proof Note that the order ν of $p(H)$ is given by Theorem 2(i). By (5), $\mu_i(p(H)) = \nu - \mu_{\nu-i}(\overline{p(H)})$, for $1 \leq i \leq \nu - 1$, and $\mu_\nu(p(H)) = 0$. Hence, by taking into account (6) and using Lemma 5, we get the result. Observe that $\{\nu^{[c_0]}\} = c_0\{\nu - 0\}$ and $\{(\nu - \mu_i(\bar{H}))^{[c_1]} : 1 \leq i \leq n\} = c_1(\nu - \sigma_L(\bar{H}))$. □

It follows that the largest Laplacian eigenvalue of $p(H)$ is ν with multiplicity at least $-1 + \sum_{k=1}^{d} c_k$. For the algebraic connectivity of $p(H)$, we deduce the following corollary.

Corollary 4. *The algebraic connectivity of* $p(H)$ *is given by*

$$\mu_{\nu-1}(p(H)) = \nu - n^{d-1}(n - \mu_{n-1}(H)).$$

Proof Observe that $\mu_j(H) \leq n$, for $1 \leq j \leq n-1$ and $\deg_H(v) \leq n-1$, for $v \in V(H)$. Hence, for $2 \leq k \leq d$, $1 \leq j \leq n-1$, $1 \leq i \leq k-1$, and $(j_i, \ldots, j_{k-1}) \in V_{H^{k-i}}$, we obtain

$$\begin{aligned} 0 \leq \nu - n^{k-1}(n - \mu_j(H)) &\leq \nu - n^{i-1}(n - \mu_j(H)) \\ &\leq \nu - n^{i-1}(n - \mu_j(H)) + \sum_{s=i}^{k-1} n^s(n - 1 - \deg_H(j_s)). \end{aligned}$$

Taking that $k = d$ and $j = n-1$, we obtain the eigenvalue $\nu - n^{d-1}(n - \mu_{n-1}(H))$ which is not greater than any other eigenvalue except 0. Therefore, since the eigenvalue 0 has multiplicity 1, the algebraic connectivity of $p(H)$ is given by the above formula. □

Example 2. Let H be the complete bipartite graph $K_{2,3}$ and consider the lexicographic polynomial

$$p(H) = 3 \cdot H^0 + 2 \cdot H + H^3$$

of order 138. Since the L-spectrum of $K_{2,3}$ is $\sigma_L(K_{2,3}) = \{5, 3, 2^{[2]}, 0\}$, applying Theorem 5 we obtain

$$\begin{aligned} \sigma_L(p(H)) = \Big\{ & 138^{[8]}, 136^{[2]}, 135^{[4]}, 133^{[2]}, 128^{[3]}, 123^{[2]}, 118^{[7]}, 113^{[6]}, 108^{[4]}, 106^{[4]}, \\ & 105^{[8]}, 103^{[6]}, 101^{[6]}, 100^{[12]}, 88, 83^{[6]}, 81^{[6]}, 80^{[12]}, 78^{[9]}, 76^{[9]}, 75^{[18]}, 63^{[2]}, 0 \Big\}. \end{aligned}$$

6. CONCLUDING REMARKS

We finish the paper with a few remarks. Recall that cospectral graphs are non-isomorphic but share the same spectrum and that the spectrum of an integral graph consists entirely of integers.

Remark 3. *The reader may observe that Theorems* 3 *and* 5 *can be used to construct infinite families of cospectral graphs. Namely, if* G *and* H *are regular and cospectral, then (by Theorem* 3*) their lexicographic polynomials* $p(H)$ *and* $p(G)$ *are also cospectral. Similarly, if we remove the connectivity and regularity conditions and consider the* L*-spectrum instead, then Theorem* 5 *leads to the same conclusion.*

For example, it is known that the smallest pair of (connected) cospectral regular graphs have 10 vertices each [**6**]. So, we may use these graphs as a starting point to construct lexicographic polynomials that represent cospectral graphs. Observe that the order of the corresponding lexicographic polynomials increases dramatically. For example, if H is one of mentioned regular graphs (with 10 vertices), then $\sum_{k=0}^{5} k \cdot H^k$ is a graph with 543 210 vertices.

Remark 4. *Another application concerns integral graphs. Note that with respect to the adjacency matrix, the lexicographic polynomial of an integral graph need not be integral (see Example* 1*). In order to obtain an integral graph, the coefficients of the lexicographic polynomial need to be chosen carefully. On the other hand, for the* L*-spectrum, the lexicographic polynomial of an integer graph always results in another integral graph, as illustrated in Example* 2.

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