



## Lower bounds for the algebraic connectivity of graphs with specified subgraphs

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### Abstract

The second smallest eigenvalue of the Laplacian matrix of a graph  $G$  is called the algebraic connectivity and denoted by  $a(G)$ . We prove that

$$a(G) > \frac{\pi^2}{3} \left( p \frac{12\bar{g}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\bar{g}(n_1, n_2, \dots, n_p)^4} + 4(q-p) \frac{3\bar{g}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\bar{g}(n_{p+1}, n_{p+2}, \dots, n_q)^4} \right),$$

holds for every non-trivial graph  $G$  which contains edge-disjoint spanning subgraphs  $G_1, G_2, \dots, G_q$  such that, for  $1 \leq i \leq p$ ,  $a(G_i) \geq a(P_{n_i})$ , with  $n_i \geq 2$ , and, for  $p+1 \leq i \leq q$ ,  $a(G_i) \geq a(C_{n_i})$ , where  $P_{n_i}$  and  $C_{n_i}$  denote the path and the cycle of the corresponding order, respectively, and  $\bar{g}$  denotes the geometric mean of given arguments. Among certain consequences, we emphasize the following lower bound

$$a(G) > \pi^2 \frac{12(4q-3p)n^2 - (16q-15p)\pi^2}{12n^4},$$

referring to  $G$  which has  $n$  ( $n \geq 2$ ) vertices and contains  $p$  Hamiltonian paths and  $q-p$  Hamiltonian cycles, such that all of them are edge-disjoint. We also discuss the quality of the obtained lower bounds.

*Keywords:* edge-disjoint subgraphs, Laplacian matrix, algebraic connectivity, geometric mean, Hamiltonian cycle

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### 1. Introduction

The *Laplacian* of a graph  $G$  is the positive semidefinite matrix  $L(G) = D(G) - A(G)$ , where  $D(G)$  is the diagonal matrix of vertex degrees and  $A(G)$  is the standard adjacency matrix. Among all eigenvalues of the Laplacian of a graph, one of the most popular is the second smallest called, by Fiedler [5], the *algebraic connectivity* of a graph. The algebraic connectivity is usually denoted by  $a(G)$ . Its significance is due to the fact that it measures (to a certain extent) how well a graph is connected. For example, a graph  $G$  is connected if and only if  $a(G) > 0$ .

The number of vertices (also known as the *order*) and the number of edges of a graph  $G$  are denoted by  $n$  and  $m$  (or  $m(G)$ ), respectively. We also use  $d$  for the diameter of a graph. A path and a cycle of order  $n$  are denoted by  $P_n$  and  $C_n$ , respectively. A graph is *Hamiltonian* if it contains a spanning subgraph which is a cycle, while every such cycle is referred to as a *Hamiltonian cycle*. Similarly, every spanning path is referred to as a *Hamiltonian path*.

There is a significantly large number of bounds for the algebraic connectivity expressed in terms of other graph invariants. One of them is a classical result of Mohar [8] stating that

$$a(G) \geq \frac{4}{dn}, \tag{1}$$

where, as said above,  $d$  is the diameter of  $G$ . Some others can be found in [1, 4, 10]. In this study we obtain a lower bound for  $a(G)$  which relies on the assumption that  $G$  contains edge-disjoint spanning subgraphs such that the algebraic connectivity of each of them is not less than the algebraic connectivity of either a fixed path or a fixed cycle. This result yields the lower bound for  $a(G)$  expressed in terms of orders of the longest paths or cycles contained in the corresponding spanning subgraphs. In particular, we establish a lower bound when  $G$  contains the set of edge-disjoint Hamiltonian paths and cycles.

Our contribution is reported in the forthcoming sections. Precisely, theoretical results are given in Section 2, a concluding discussion is given in Section 3, while in the Appendix we observe the existence of an upper bound for the algebraic connectivity (which is implicitly proved in [2]).

### 2. Results

We use the following lemma referred to Fiedler.

**Lemma 2.1.** [5] *Let  $G_1, G_2, \dots, G_k$  be edge-disjoint spanning subgraphs of a non-trivial signed graph  $G$  such that  $m(G) = \sum_{i=1}^k m(G_i)$ . Then*

$$a(G) \geq \sum_{i=1}^k a(G_i).$$

We also use the following limit point without reference:

$$\lim_{x \rightarrow 0} \left( \frac{\sum_{i=1}^k t_i^x}{k} \right)^{\frac{1}{x}} = \left( \prod_{i=1}^k t_i \right)^{\frac{1}{k}}, \tag{2}$$

for positive  $t_1, t_2, \dots, t_k$ .

**Theorem 2.1.** Assume that a graph  $G$  with  $n$  ( $n \geq 2$ ) vertices contains edge-disjoint spanning subgraphs  $G_1, G_2, \dots, G_q$  such that for  $1 \leq i \leq p$  it holds  $a(G_i) \geq a(P_{n_i})$  with  $n_i \geq 2$  and for  $p + 1 \leq i \leq q$  it holds  $a(G_i) \geq a(C_{n_i})$ . Then

$$a(G) > \frac{\pi^2}{3} \left( p \frac{12\bar{g}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\bar{g}(n_1, n_2, \dots, n_p)^4} + 4(q - p) \frac{3\bar{g}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\bar{g}(n_{p+1}, n_{p+2}, \dots, n_q)^4} \right), \quad (3)$$

where  $\bar{g}$  denotes the geometric mean of given arguments.

*Proof.* By Lemma 2.1,  $a(G) \geq \sum_{i=1}^q a(G_i)$ , i.e.,  $a(G) \geq \sum_{i=1}^p a(P_{n_i}) + \sum_{i=p+1}^q a(C_{n_i})$ . It holds  $a(P_{n_i}) = 2(1 - \cos(\frac{\pi}{n_i}))$  and  $a(C_{n_i}) = 2(1 - \cos(\frac{2\pi}{n_i}))$ ; see, for example, [1].

Using the Taylor series, we get

$$a(P_{n_i}) > 2 \left( 1 - 1 + \frac{\pi^2}{2n_i^2} - \frac{\pi^4}{24n_i^4} \right) = \frac{\pi^2}{12n_i^4} (12n_i^2 - \pi^2)$$

and

$$a(C_{n_i}) > 2 \left( 1 - 1 + \frac{4\pi^2}{2n_i^2} - \frac{16\pi^4}{24n_i^4} \right) = \frac{4\pi^2}{3n_i^4} (3n_i^2 - \pi^2)$$

that gives

$$a(G) > \frac{\pi^2}{3} \left( \frac{1}{4} \sum_{i=1}^p \frac{12n_i^2 - \pi^2}{n_i^4} + 4 \sum_{i=p+1}^q \frac{3n_i^2 - \pi^2}{n_i^4} \right). \quad (4)$$

We consider the first sum of (4). For  $\alpha \geq 2$ , we define the function

$$f_\alpha(x) = \frac{12x^\alpha - \pi^2}{x^{2\alpha}}.$$

It holds  $f''_\alpha(x) = \frac{2\alpha}{x^{2(\alpha+1)}} (6(\alpha + 1)x^\alpha - \pi^2(2\alpha + 1))$ , and so, for  $x \geq 2$ ,  $f_\alpha$  is convex. Using the Jensen's inequality, we get

$$\sum_{i=1}^p \frac{12n_i^2 - \pi^2}{n_i^4} \geq p f_\alpha \left( \frac{\sum_{i=1}^p n_i^{2/\alpha}}{p} \right) = p \frac{12 \left( \frac{\sum_{i=1}^p n_i^{2/\alpha}}{p} \right)^\alpha - \pi^2}{\left( \frac{\sum_{i=1}^p n_i^{2/\alpha}}{p} \right)^{2\alpha}}.$$

If  $\alpha \rightarrow \infty$ , by (2), we have

$$\sum_{i=1}^p \frac{12n_i^2 - \pi^2}{n_i^4} \geq p \frac{12\bar{g}(n_1, n_2, \dots, n_p)^2 - \pi^2}{\bar{g}(n_1, n_2, \dots, n_p)^4}. \quad (5)$$

The second sum of (4) is considered in a similar way. For  $\alpha \geq 3$ , we define the function

$$h_\alpha(x) = \frac{3x^\alpha - \pi^2}{x^{2\alpha}},$$

which is convex for  $x \geq 3$  (as  $h''_\alpha(x) = \frac{a}{x^{2(\alpha+1)}}(3(\alpha+1)x^\alpha - 2\pi^2(2\alpha+1))$ ). This leads to

$$\sum_{i=p+1}^q \frac{3n_i^2 - \pi^2}{n_i^4} \geq (q-p)h_\alpha\left(\frac{\sum_{i=p+1}^q n_i^{2/\alpha}}{q-p}\right) = (q-p) \frac{3\left(\frac{\sum_{i=p+1}^q n_i^{2/\alpha}}{q-p}\right)^\alpha - \pi^2}{\left(\frac{\sum_{i=p+1}^q n_i^{2/\alpha}}{q-p}\right)^{2\alpha}}.$$

Letting  $\alpha \rightarrow \infty$ , we get

$$\sum_{i=p+1}^q \frac{3n_i^2 - \pi^2}{n_i^4} \geq (q-p) \frac{3\bar{g}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\bar{g}(n_{p+1}, n_{p+2}, \dots, n_q)^4}. \tag{6}$$

The inequality (4), in conjunction with (5) and (6), gives (3). □

Here are some consequences.

**Corollary 2.1.** *Under the assumptions of Theorem 2.1, we have*

$$a(G) > \frac{\pi^2}{3} \left( p \frac{12\bar{a}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\bar{a}(n_1, n_2, \dots, n_p)^4} + 4(q-p) \frac{3\bar{a}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\bar{a}(n_{p+1}, n_{p+2}, \dots, n_q)^4} \right), \tag{7}$$

where  $\bar{a}$  denotes the arithmetic mean of given arguments.

*Proof.* The function  $\frac{12x^2 - \pi^2}{4x^2}$  decreases for  $x \geq 2$ , and so

$$\frac{12\bar{g}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\bar{g}(n_1, n_2, \dots, n_p)^4} \geq \frac{12\bar{a}(n_1, n_2, \dots, n_p)^2 - \pi^2}{4\bar{a}(n_1, n_2, \dots, n_p)^4}.$$

Similarly, as  $\frac{3x^2 - \pi^2}{x^2}$  decreases for  $x \geq 3$ , we have

$$\frac{3\bar{g}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\bar{g}(n_{p+1}, n_{p+2}, \dots, n_q)^4} \geq \frac{3\bar{a}(n_{p+1}, n_{p+2}, \dots, n_q)^2 - \pi^2}{\bar{a}(n_{p+1}, n_{p+2}, \dots, n_q)^4},$$

and the proof follows. □

**Corollary 2.2.** *Under the assumptions of Theorem 2.1, we have*

$$a(G) > q\pi^2 \frac{12\bar{g}(n_1, n_2, \dots, n_q)^2 - \pi^2}{12\bar{g}(n_1, n_2, \dots, n_q)^4} \geq q\pi^2 \frac{12\bar{a}(n_1, n_2, \dots, n_q)^2 - \pi^2}{12\bar{a}(n_1, n_2, \dots, n_q)^4}, \tag{8}$$

where  $\bar{g}$  and  $\bar{a}$  denote the geometric mean and the arithmetic mean of given arguments, respectively.

*Proof.* In the notation of Theorem 2.1, since  $a(C_{n_i}) > a(P_{n_i})$ , we have  $a(G_i) \geq a(P_{n_i})$ , for  $1 \leq i \leq q$ . The first inequality follows by setting  $p = q$  in (3), and then the second follows by the previous corollary. □

We proceed with the following particular result.

**Theorem 2.2.** *If a non-trivial graph  $G$  contains  $p$  Hamiltonian paths and  $q-p$  Hamiltonian cycles, such that all of them are edge disjoint, then*

$$a(G) > \pi^2 \frac{12(4q - 3p)n^2 - (16q - 15p)\pi^2}{12n^4}. \tag{9}$$

*Proof.* Obviously,  $G$  contains edge-disjoint spanning subgraphs  $G_1, G_2, \dots, G_q$  such that the first  $p$  of them contain a Hamiltonian path and the remaining ones contain a Hamiltonian cycle. By Lemma 2.1, the algebraic connectivity of  $G_i$  is at least the algebraic connectivity of its spanning subgraph, i.e., all the assumptions of Theorem 2.1 are satisfied (with  $n_i = n$ , for  $1 \leq i \leq q$ ). By (3), we compute

$$a(G) > \frac{\pi^2}{3} \left( p \frac{12n^2 - \pi^2}{4n^4} + 4(q - p) \frac{3n^2 - \pi^2}{n^4} \right),$$

giving the desired inequality. □

Since, for a connected graph  $G$ , we have  $a(G) \geq 2\epsilon(1 - \cos \frac{\pi}{n})$  (see [5]), where  $\epsilon$  denotes the edge connectivity of  $G$ , it follows that Theorem 2.1 can be applied to any connected non-trivial graph with itself in the role of the unique spanning subgraph. Here is another criterion concerning graphs with small diameter.

**Theorem 2.3.** *If a connected graph  $G$  with  $n$  ( $n \geq 2$ ) vertices and diameter  $d$  contains a path  $P_k$  (resp. a cycle  $C_k$ ) such that  $4k^2 \geq dn\pi^2$  (resp.  $k^2 \geq dn\pi^2$ ), then  $a(G) > a(P_k)$  (resp.  $a(G) > a(C_k)$ ).*

*Proof.* We use the inequality (1). Considering the existence of a path  $P_k$ , we get

$$a(G) \geq \frac{4}{dn} \geq \frac{4}{\frac{4k^2}{\pi^2}} = \frac{\pi^2}{k^2} = 2 \left( 1 - 1 + \frac{\pi^2}{2k^2} \right) > 2 \left( 1 - \cos \frac{\pi}{k} \right).$$

The existence of a cycle satisfying the assumption of the theorem is considered in the same way. □

### 3. Remarks

The bound (3) and its consequences (7)–(9) are always non-trivial, in the sense that they are never negative. An easy consequence of (9) is the following lower bound

$$a(G) > 4q\pi^2 \frac{3n^2 - \pi^2}{3n^4}, \tag{10}$$

where  $q$  stands for the number of edge-disjoint Hamiltonian cycles. In general, the bound (10) is incomparable with (1), but it gives a better estimate whenever

$$q \geq \frac{3n^3}{d\pi^2(n^2 - \pi^2)}. \tag{11}$$

In particular, this occurs for every Hamiltonian graph with  $d \geq \frac{3n^3}{\pi^2(3n^2 - \pi^2)}$ , as then the right hand side of (11) is at most 1; this lower bound for  $d$  is asymptotically  $n/\pi^2$ .

**Example 1.** Consider the graph  $G$  obtained by inserting an edge between every pair of vertices at distance 2 of a cycle  $C_{2k+1}$ , for  $k \geq 2$ . Obviously,  $G$  has exactly 2 edge-disjoint Hamiltonian cycles, and thus due to (10) we have  $a(G) > 8\pi^2 \frac{3(2k+1)^2 - \pi^2}{3(2k+1)^4}$ . Say, for  $k = 4$ , we get  $2.12 \approx a(G) > 0.94$ .

As the right hand side of (10) increases with the number of edge-disjoint Hamiltonian cycles, it would be natural to consider it in conjunction with a lower bound for the number of such cycles. In this context, we recall that Nash-Williams proved that the assumptions of the well-known Dirac’s theorem guarantee the existence of many edge-disjoint Hamiltonian cycles. Precisely, every graph with  $n$  vertices and minimum vertex degree at least  $n/2$  contains at least  $\lfloor 5n/224 \rfloor$  edge-disjoint Hamiltonian cycles [9]. It is conjectured in the same reference that every  $r$ -regular graph with at most  $2r$  vertices contains  $r/2$  Hamiltonian cycles. This conjecture is still open; an approximate version stating that every  $r$ -regular graph with  $n$  ( $14 \leq n \leq 2r + 1$ ) vertices contains  $\lfloor (3r - n + 1)/6 \rfloor$  edge-disjoint Hamiltonian cycles is proved by Jackson [7]. For some asymptotic results, we refer to Christofides, Kühn and Osthus [3]. Particular constructions of arbitrarily large graphs with a specified number of Hamiltonian cycles can be found in Haythorpe’s [6].

### Appendix

We recall an interesting upper bound, obtained by Bollobás and Nikiforov [2], for the sum of the  $k - 1$  least eigenvalues of a Hermitian matrix. Namely, if  $N_1 \sqcup N_2 \cdots \sqcup N_k$  is a partition of a Hermitian matrix  $M = (m_{ij})$  with eigenvalues  $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ , then

$$\sum_{p=n-k+2}^n \nu_p \leq \sum_{p=1}^k \frac{1}{|N_p|} \sum_{(i,j): i,j \in N_p} m_{ij} - \frac{1}{n} \text{sum}(M), \tag{12}$$

where  $\text{sum}(M)$  denotes the sum of the entries of  $M$ .

By considering the Laplacian matrix of a graph in the role of  $M$  and inserting  $k = 3$  in (12), we get

$$a(G) \leq \sum_{p=1}^3 \frac{c(N_p)}{|N_p|}, \tag{13}$$

where, clearly  $N_1 \sqcup N_2 \sqcup N_3$  is a vertex set partition, while  $c(N_p)$  denotes the *cut* of  $N_p$ , i.e., the number of edges with exactly one end in  $N_p$ . Indeed, if  $L = (l_{ij})$  is the Laplacian matrix, then  $\sum_{(i,j): i,j \in N_p} l_{ij} = c(N_p)$  and  $\text{sum}(L) = 0$ , so we get (13). This upper bound can be used to estimate the algebraic connectivity of graphs with given tripartition of a vertex set. For example, if  $G$  contains at least two cut-edges, then we have

$$a(G) \leq \frac{1}{|N_1|} + \frac{2}{|N_2|} + \frac{1}{|N_3|},$$

where cut-edges are located between  $N_1$  and  $N_2$ , and  $N_2$  and  $N_3$ .

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## References

- [1] N. M. M. de Abreu, Old and new results on algebraic connectivity of graphs, *Linear Algebra Appl.*, **423** (2007), 53–73.
- [2] B. Bollobás and V. Nikiforov, Graphs and Hermitian matrices: eigenvalue interlacing. *Discrete Math.* **289** (2004), 119–127.
- [3] D. Christofides, D. Kühn and D. Osthus, Edge-disjoint Hamilton cycles in graphs, *J. Combin. Theory B*, **102** (2012), 1035–1060.
- [4] K. Ch. Das, Proof of conjectures involving algebraic connectivity of graphs, *Linear Algebra Appl.*, **438** (2013), 3291–3302.
- [5] M. Fiedler, Algebraic connectivity of graphs, *Czech. Math. J.*, **23** (1973), 298–305.
- [6] M. Haythorpe, Constructing arbitrarily large graphs with a specified number of Hamiltonian cycles, *Electron. J. Graph Theory Appl.*, **4** (1) (2016), 18–25.
- [7] B. Jackson, Edge-disjoint Hamilton cycles in regular graphs of large degree, *J. London Math. Soc.*, **19** (1979), 13–16.
- [8] B. Mohar, Eigenvalues, diameter, and mean distance in graphs, *Graph Combinator.*, **7** (1991), 53–64.
- [9] C. St. J. A. Nash-Williams, Edge-disjoint Hamiltonian circuits in graphs with vertices of large valency, in: L. Mirsky (Ed.), *Studies in Pure Mathematics*, Academic Press, 1971, pp. 157–183.
- [10] A. A. Rad, M. Jalili and M. Hasler, A lower bound for algebraic connectivity based on the connection-graph-stability method, *Linear Algebra Appl.*, **435** (2011), 186–192.