On star complement technique

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Abstract

The star complement technique is a spectral tool developed for constructing larger graphs from their induced subgraphs, called star complements. Here we give a short description of this technique; for more details, we refer the reader to [4, 6, 9].

We begin with a general concept supported by theoretical results. This concept assumes that the eigenvalue we are dealing with (see below) is distinct from 0 and -1.

We consider only simple graphs, that is finite, undirected graphs without loops or multiple edges. If G is such a graph with vertex set $V_G = \{1, 2, \ldots, n\}$, the *adjacency matrix* of G is the $n \times n$ matrix $A_G = (a_{ij})$, where $a_{ij} = 1$ if there is an edge between the vertices i and j, and 0 otherwise. The *eigenvalues* of G, denoted by

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n,$$

are just the eigenvalues of A_G. Note, the eigenvalues of G are real and do not depend on vertex labelling. Additionally, for connected graphs $\lambda_1 > \lambda_2$ holds. The *characteristic polynomial* of G is the characteristic polynomial of its adjacency matrix, so $P_G(\lambda) = \det(\lambda I - A_G)$. For more details on graph spectra, see [3].

If μ is an eigenvalue of G of multiplicity k, then a *star set* for μ in G is a set X of k vertices taken from G such that μ is not an eigenvalue of $G - X$. The graph $H = G - X$ is called the *star complement* for μ in G (or a *μ–basic subgraph* of G in [7]). (Star sets and star complements exist for any eigenvalue and any graph; they need not be unique.) The H–neighbourhoods of vertices in X can be shown to be non–empty and distinct, provided $\mu \notin \{-1, 0\}$ (see [5], Chapter 7). If $t = |V_H|$, then $|X| \leq \binom{t}{2}$ (see [1]) and this bound is best possible.

It can be proved that if Y is a proper subset of X then $X - Y$ is a star set for μ in $G - Y$, and therefore H is a star complement for μ in $G - Y$. If G has star complement H for μ, and G is not a proper induced subgraph of some other graph with star complement H for μ, then G is a *maximal graph* with star complement H for μ, or it is an *H–maximal graph* for μ. It follows that there are only finitely many such maximal graphs, provided $\mu \notin \{-1, 0\}$. In general, there will be only several maximal graphs, possibly of different orders, but sometimes there is a unique maximal graph (if so, this graph is characterized by its star complement for μ).

We now mention some results from literature (they are taken from [4, 5, 6]).

The following result is known as the Reconstruction Theorem (see, for example, [5, Theorems 7.4.1 and 7.4.4]).

Theorem 1 *Let G be a graph with adjacency matrix*

$$\begin{pmatrix} A_X & B^T \\ B & C \end{pmatrix},$$

where A_X is the adjacency matrix of the subgraph induced by the vertex set X. Then X is a star set for μ if and only if μ is not an eigenvalue of C and $\mu I - A_X = B^T(\mu I - C)^{-1}B$.

Hence, if μ, C and B are fixed then A_X is uniquely determined. In other words, given the eigenvalue μ, a star complement H for μ, and the H–neighbourhoods of vertices in the star set X, the graph G is uniquely determined. In the light of these facts, we may next ask to what extent G is determined only by H and μ. bearing in mind the previous observations, it is sufficient to consider graphs G which are H–maximal for μ.

Following [2, 8, 10], we list some notation and terminology. Given a graph H, a subset U of $V(H)$ and a vertex u not in $V(H)$, denote by $H(U)$ the graph obtained from H by joining u to all vertices of U. We will say that u (resp. U, $H(U)$) is a *good vertex* (*good set*, *good extension*) for μ and H, if μ is an eigenvalue of $H(U)$ but is not an eigenvalue of H. By Theorem 1, a vertex u and a subset U are good if and only if $\mathbf{b}_u^T(\mu I - C)^{-1}\mathbf{b}_u = \mu$, where $\mathbf{b}_u$ is the characteristic vector of U (with respect to $V(H)$) and C is the adjacency matrix of H. Assume now that U_1 and U_2 are not necessarily good sets corresponding to vertices u_1 and u_2, respectively. Let $H(U_1, U_2; 0)$ and $H(U_1, U_2; 1)$ be the graphs obtained by adding to H both vertices, u_1 and u_2, so that they are non–adjacent in the former graph, while adjacent in the latter graph. We say that u_1 and u_2 are *good partners* and that U_1 and U_2 are *compatible sets* if μ is an eigenvalue of multiplicity two either in $H(U_1, U_2; 0)$ or in $H(U_1, U_2; 1)$. (Note, if $\mu \notin \{-1, 0\}$, any good set is non–empty, any two of them if corresponding to compatible sets are distinct; see [5], Proposition 7.6.2.) By Theorem 1, two vertices u_1 and u_2 are good partners (or two sets U_1 and U_2 are compatible) if and only if $\mathbf{b}_{u_1}^T(\mu I - C)^{-1}\mathbf{b}_{u_2} \in \{-1, 0\}$, where $\mathbf{b}_{u_1}$ and $\mathbf{b}_{u_2}$ are defined above. In addition, it follows (again by Theorem 1) that any vertex set X in which all vertices are good, both individually and in pairs, gives rise to a *good extension*, say G, in which X can be viewed as a star set for μ, while H as the corresponding star complement.

The previous considerations show us how we can introduce a technique, also called the *star complement technique*, for finding (or constructing) graphs with certain spectral properties. In this context the graphs we are interested in should have some prescribed eigenvalue usually of a very large multiplicity. If G is a graph in which μ is an eigenvalue of multiplicity $k > 1$, then G is a good (k–vertex) extension of some of its star complements, say H (in particular, G is H–maximal for μ). The *star complement* technique consists of the following: In order to find H–maximal graphs for μ ($\neq -1, 0$), we form an *extendability graph* whose vertices are good vertices for μ and H, and add an edge between two good vertices whenever they are good partners. Now it is easy to see that the search for maximal extensions is reduced to the search for maximal cliques in the extendability graph (see [4, 6]). Of course, among H–maximal graphs some of them can be mutually isomorphic. So, we determine how many different isomorphism classes they belong to. An explicit algorithm is given in [9, p. 101].

The previous discussion excludes the possibility $\mu \in \{0, -1\}$. This is a natural restriction since if μ takes any of these values, then for every H, there is an infinite family of the corresponding μ-extensions. To see this, it is sufficient to observe that in this case X of the Reconstruction Theorem may contain an arbitrary number of vertices sharing the same neighbourhood in H. for $\mu = 0$ (resp. $\mu = -1$) these vertices are mutually non-adjacent (adjacent).

Mutually non-adjacent (adjacent) vertices u, v, such that $N(u) = N(v)$ ($N(u) \cup \{u\} = N(v) \cup \{v\}$) are called *twins* (*co-twins*). If we restrict ourselves to *basic μ-extensions*; i.e., μ-extensions without twin vertices for $\mu = 0$ (resp. co-twin vertices for $\mu = -1$), then there are only finitely many such extensions and the previous method is applied with slight modifications. This is implemented in version 2.2 of SCL, as the previous ones are restricted to $\mu \in \{0, -1\}$. We note that the theoretical approach does not exclude neither twin nor co-twin vertices, and so this restriction is adapted only for $\mu \in \{0, -1\}$. It results in the set of maximal basic extensions.

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